Optimal Control of Temperature in Fluid Flow

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Abstract

This paper presents an optimal control problem of temperature using the adjoint equation method of an optimal control theory and the finite element method. The three dimensional model is used to analyze the thermal fluid flow, which is described by the incompressible Navier-Stokes equation and the energy equation. The formulation is based on the optimal control theory in which the performance function is expressed by the computed and target temperatures. The optimal value can be obtained by minimizing the performance function. The fractional step method is applied to solve the incompressible Navier-Stokes equation. The weighted gradient method is employed as a minimization technique.


1 Introduction

The thermal fluid flow is a phenomenon generated by non-homogeneity of temperature. The thermal fluid flow is divided into natural fluid flow and forced fluid flow. The natural fluid flow is caused by the difference of the specific gravity between warmed fluid and cool fluid. Forced fluid flow is compulsorily caused by fan or other items. In recent years, the optimal control of thermal fluid flow is used in various places in our life. For instance, the optimal control of the thermal fluid flow is applied to air-conditioning, heating, refrigerator and bath and so on. Therefore, the optimal control of thermal fluid flow is indispensable and very important problem. Especially, the optimal control in the natural fluid flow is one of the most-attained recent problem. This is because the control brings low cost and good for the ecology. Floor heating system and temperature management of grass field in the football stadium are good examples. Therefore, this research covers the problem in the natural fluid flow.

The purpose of this study is the control of temperature in the natural fluid flow. The temperature is controlled using the three dimensional field as a computational domain. The incompressible Navier-Stokes and the energy equations are employed for the state equation. In the optimal control theory, the control variable that makes the optimal state can be obtained by minimizing the performance function. The performance function is composed of the square sum of difference between computed and target temperatures. The control variable can be converged to the target temperature in case that the performance function is minimized. The weighted gradient method is applied as the minimization technique. The fractional step method is applied to solve the Navier-Stokes equation. The fractional step method in this research uses the intermediate velocity that does not completely satisfy the continuity equation. The Crank-Nicolson method and the Galerkin method expanded by the mixed interpolation are applied to temporal and spatial discretization, respectively. The stabilized bubble function is applied to velocity and temperature fields. The linear interpolation is utilized to pressure field.

This study presents an analysis and a control problem of thermal fluid flow, in which two numerical studies are carried out. In numerical study 1, thermal fluid flow is analyzed as the three dimensional model. In numerical study 2, the optimal control of thermal fluid flow is performed out in the three dimensional model.
2 State Equation

The thermal fluid flow is described by the incompressible Navier-Stokes equation and the energy equation. Therefore, these equations are employed for the state equation. The incompressible Navier-Stokes equation is applied to the fluid field. The energy equation is applied to the temperature field. In the incompressible flow, density of fluid is assumed constant. The Navier-Stokes equation and energy equation are written as follows:

\[ \dot{u}_i + u_j u_{i,j} + p_i - \nu(u_{i,jj} + u_{j,ij}) = f_i \theta \quad \text{in} \quad \Omega, \]  
\[ u_{i,i} = 0 \quad \text{in} \quad \Omega, \]  
\[ \dot{\theta} + u_j \theta_{,j} - \kappa \theta_{,jj} = 0 \quad \text{in} \quad \Omega, \]

where \( \nu, \kappa \) and \( f_i \) are written as follows:

\[ \nu = Pr, \quad f_i = PrRa, \]  
where \( u_i, p, \theta \) and \( f_i \) are the velocity, the pressure, the temperature and the gravitational acceleration, \( \nu \) and \( \kappa \) are the kinematic viscosity coefficient and the diffusion coefficient, respectively. The Rayleigh number and the Prandtl number are denoted by \( Ra \) and \( Pr \), respectively. The initial conditions are given as follows:

\[ u_i(t_0) = \hat{u}_i^0, \]  
\[ \theta(t_0) = \hat{\theta}^0, \]  

where \( \hat{u}_i^0 \) and \( \hat{\theta}^0 \) are the initial known value for velocity and temperature, respectively. The boundary conditions are given as follows:

\[ u_i = \hat{u}_i \quad \text{on} \quad \Gamma_D, \]  
\[ t_i = -p\delta_{ij} + \nu(u_{i,j} + u_{j,i})n_j = \hat{t}_i \quad \text{on} \quad \Gamma_N, \]  
\[ \theta = \hat{\theta} \quad \text{on} \quad \Gamma_D, \]  
\[ b = \kappa \theta_{,i} n_i = \hat{b} \quad \text{on} \quad \Gamma_N, \]  
\[ \theta = \hat{\theta}_{\text{cont}} \quad \text{on} \quad \Gamma_C, \]

where subscripts \( D, N \) and \( C \) of \( \Gamma \) are the Dirichlet boundary, the Neumann boundary and the control boundary, respectively. The control variable on the control boundary \( \Gamma_C \) is denoted by \( \hat{\theta}_{\text{cont}} \).

3 Fractional Step Method

To solve the state equation and the adjoint equation, the Crank-Nicolson method is used for the temporal discretization. This method is capable of taking long time increment and superior in stability. Therefore, a lot of time cycles can be taken in the computation. The momentum equation, continuity equation and energy equation are expressed as follows:

\[ \frac{u^{n+1}_i - u^n_i}{\Delta t} + u_j u_{i,j}^{n+\frac{1}{2}} + p_i^{n+1} - \nu(u_{i,jj}^{n+\frac{1}{2}} + u_{j,ij}^{n+\frac{1}{2}}) = f_i \theta \quad \text{in} \quad \Omega, \]  
\[ u_{i,i}^{n+1} = 0 \quad \text{in} \quad \Omega, \]  
\[ \frac{\theta^{n+1}_i - \theta^n_i}{\Delta t} + u_j \theta_{,j}^{n+\frac{1}{2}} - \kappa \theta_{,jj}^{n+\frac{1}{2}} = 0 \quad \text{in} \quad \Omega, \]  
\[ u_i^{n+\frac{1}{2}} = \frac{1}{2}(u_i^n + u_i^{n+1}), \]  
\[ \theta_i^{n+\frac{1}{2}} = \frac{1}{2}(\theta_i^n + \theta_i^{n+1}). \]
The fractional step method is applied to solve the incompressible Navier-Stokes equation. This method is a technique that is divided into the velocity and the pressure by deriving the pressure Poisson equation. The pressure Poisson equation is derived after obtaining the intermediate velocity $\tilde{u}$.

The pressure at the previous step $p^n$ is assumed to be approximate pressure. The pressure of the momentum equation is replaced with $p^n$. Unknown velocity is replaced with the intermediate velocity $\tilde{u}$. Then, the momentum equation in Eq.(12) is as follows:

$$\frac{\tilde{u}_i^{n+1} - u_i^n}{\Delta t} + u_j \tilde{u}_{i,j} + p_i^n - \nu (\tilde{u}_{i,j} + \tilde{u}_{j,i}) = f_i \theta \quad \text{in} \quad \Omega. \quad (17)$$

The difference between Eq.(12) and Eq.(17) is taken, the following equation is obtained.

$$\frac{u_i^{n+1} - \tilde{u}_i^n}{\Delta t} + \frac{1}{2} u_j (u_{i,j}^{n+1} - u_{i,j}^n) + (p_i^{n+1} - p_i^n) - \frac{1}{2} \nu \{(u_{i,j} + u_{j,i})^{n+1} - (\tilde{u}_{i,j} + \tilde{u}_{j,i})^{n+1}\}$$

$$= f_i \theta \quad \text{in} \quad \Omega. \quad (18)$$

The pressure Poisson equation is obtained from the transpiration of Eq.(18) considering the continuity equation as follows:

$$\Delta t (p_i^{n+1} - p_i^n) = \tilde{u}_i^{n+1} \quad \text{in} \quad \Omega. \quad (19)$$

### 4 Spatial Discretization

#### 4.1 Bubble Function Interpolation

The Galerkin method is applied to the spatial discretization. The mixed interpolation based on the bubble function and linear function is used for the spatial discretization. The bubble function interpolation is applied to the velocity and the temperature fields and the linear interpolation is applied to the pressure field as follows:

For bubble function interpolation:

$$u_i = \Phi_1 u_{i1} + \Phi_2 u_{i2} + \Phi_3 u_{i3} + \Phi_4 u_{i4} + \Phi_5 \tilde{u}_{i5}, \quad (20)$$

$$\tilde{u}_{i5} = u_{i5} - \frac{1}{4} (u_{i1} + u_{i2} + u_{i3} + u_{i4}), \quad (21)$$

$$\theta = \Phi_1 \theta_1 + \Phi_2 \theta_2 + \Phi_3 \theta_3 + \Phi_4 \theta_4 + \Phi_5 \tilde{\theta}_5, \quad (22)$$

$$\tilde{\theta}_5 = \theta_5 - \frac{1}{4} (\theta_1 + \theta_2 + \theta_3 + \theta_4), \quad (23)$$

$$\Phi_1 = L_1, \quad \Phi_2 = L_2, \quad \Phi_3 = L_3, \quad \Phi_4 = L_4, \quad \Phi_5 = 256 L_1 L_2 L_3 L_4, \quad (24)$$

and for linear interpolation:

$$p = \Psi_1 p_1 + \Psi_2 p_2 + \Psi_3 p_3 + \Psi_4 p_4, \quad (25)$$

$$\Psi_1 = L_1, \quad \Psi_2 = L_2, \quad \Psi_3 = L_3, \quad \Psi_4 = L_4, \quad (26)$$
where \(u_{i1} \sim u_{i4}\) and \(\theta_1 \sim \theta_4\) are the velocity and the temperature at nodes 1 \(\sim\) 4 of each finite element. \(\Phi_\alpha (\alpha = 1 \sim 5)\) is the bubble function for the velocity and the temperature, in which \(p_1 \sim p_4\) are the pressure at nodes 1 \(\sim\) 4 of each finite element. The linear interpolation function is denoted by \(\Psi_\alpha (\alpha = 1 \sim 4)\) for pressure. Uncoordinated is expressed by \(L_i\).

### 4.2 Stabilized Form

In the bubble function, the numerical stabilization is not enough. Therefore, the stabilized parameter which determines the magnitude of streamline is used for stabilization. In equation, the stabilized parameter \(\tau_{eB}\) can be written as follows:

\[
\tau_{eB} = \frac{\langle \phi_{e,1} \rangle_{\Omega_e}^2}{\nu \| \phi_{e,j} \|_{\Omega_e}^2 A_e},
\]

where \(\Omega_e\) is element domain and

\[
\langle u, v \rangle_{e} = \int_{\Omega_e} uv d\Omega, \quad \| u \|_{\Omega_e}^2 = \int_{\Omega_e} u^2 d\Omega, \quad A_e = \int_{\Omega_e} d\Omega.
\]

The integral of bubble function is expressed as follows:

\[
\langle \phi_{e,1} \rangle_{\Omega_e} = \frac{A_e}{6}, \quad \| u_{e,j} \|_{\Omega_e}^2 = 2A_e g, \quad g = \sum_{i=1}^{2} |\Psi_{\alpha,i}|^2,
\]

where \(\nu'\) is control parameter for the stabilizing action. This value is determined to become equivalent to \(\tau_{eS}\) using the stabilized finite element method.

\[
\tau_{eS} = \left[ \left( \frac{2|u'_{i}|}{h_e} \right)^2 + \left( \frac{4\nu'}{h_e^2} \right)^2 \right]^{-\frac{1}{2}}, \quad u'_i = \sqrt{u_1 + u_2 + u_3},
\]

where \(h_e\) is an element size. In generally, the stabilized parameter in Eq.(30) is not equal to the optimal parameter in Eq.(31). Thus, the bubble function which gives the optimal viscosity satisfies the following equation.

\[
\frac{\langle \phi_{e,1} \rangle_{\Omega_e}^2}{(\nu + \nu') \| \phi_{e,j} \|_{\Omega_e}^2 A_e} = \tau_{eS}.
\]

Eq.(32) adds stabilized operator control term only of the barycenter point to the equation of motion as follows:

\[
\sum_{e=1}^{N_e} \nu' \| \phi_{e,j} \|_{\Omega_e}^2 b_e,
\]

where \(N_e\) and \(b_e\) are the total number of element and the barycenter point. In the energy equation, \(\nu\) is replaced with \(\kappa\).
5 Optimal Control Theory

5.1 Performance Function

In the optimal control theory, control variable can be obtained by minimizing performance function. The performance function $J$ is composed of the square sum of difference between computed and target temperatures. The control variable are computed to minimize the performance function under the state equation and boundary conditions. Minimizing the performance function means that computed temperature should be as close as possible to the objective temperature. The performance function is written as follows:

$$J = \frac{1}{2} \int_{t_0}^{t_f} \int_{\Omega} (\theta - \theta_{obj})^T Q(\theta - \theta_{obj}) d\Omega dt,$$

where $\theta$, $\theta_{obj}$ and $Q$ are the computed temperature, the target temperature at the object point and the weighting diagonal matrix. Superscript $T$ means transpose.

5.2 Adjoint Equation

The Lagrange multiplier method is applied to minimize the performance function. The Lagrange multiplier method is used for the minimization problem with constraints. The extended performance function $J^*$ is expressed as follows:

$$J^* = J - \int_{t_0}^{t_f} \int_{\Omega} u_i^* (u_j + u_{i,j} + p_i - \nu(u_{i,jj} + u_{j,ij}) - f_i \theta) d\Omega dt$$

$$- \int_{t_0}^{t_f} \int_{\Omega} p_i^* (u_{i,i}) d\Omega dt$$

$$- \int_{t_0}^{t_f} \int_{\Omega} \theta^* (\dot{\theta} + u_j \theta_j - \kappa \theta_{jj}) d\Omega dt,$$

where $u_i^*$, $p^*$ and $\theta^*$ denote the Lagrange multiplier for velocity, pressure and temperature, respectively. The extended performance function $J^*$ is divided into the Hamiltonian $H$ and the time differentiation term as follows:

$$J^* = \int_{t_0}^{t_f} \int_{\Omega} \{H + (u_i^* T \dot{u}_i + \theta^* T \dot{\theta})\} d\Omega dt,$$

where the Hamiltonian $H$ is defined as follows:

$$H = \frac{1}{2} (\theta - \theta_{obj})^T Q(\theta - \theta_{obj})$$

$$- u_i^* (u_j + u_{i,j} + p_i - \nu(u_{i,jj} + u_{j,ij}) - f_i \theta) - p_i^* (u_{i,i}) - \theta^* (\dot{\theta} + u_j \theta_j - \kappa \theta_{jj}).$$

The stationary condition is needed to minimize performance function. The first variation of $J^*$ should be zero as follows,

$$\delta J^* = 0.$$

Therefore, the adjoint equation and terminal conditions are obtained. The adjoint equation and the terminal conditions are written as follows:

$$u_i^* = \frac{\partial H}{\partial a}, \quad \frac{\partial H}{\partial p} = 0, \quad \dot{\theta}^* = \frac{\partial H}{\partial \theta},$$

$$u_i^*(t_f) = 0, \quad p^*(t_f) = 0, \quad \theta^*(t_f) = 0.$$
6 Minimization Technique

6.1 Weighted Gradient Method

The weighted gradient method is applied to the minimization technique. This method is a technique that control variable are renewed by the stationary condition of a modified performance function. The modified performance function $K^{(l)}$ to which the penalty term is added is defined as follows:

$$K^{(l)} = J^* + \frac{1}{2}(\bar{\theta}^{(l+1)}(t) - \bar{\theta}^{(l)}(t))^T W^{(l)}(\bar{\theta}^{(l+1)}(t) - \bar{\theta}^{(l)}(t)),$$

where $\bar{\theta}(t)$, $W^{(l)}$ and $l$ are control variable, weighting diagonal matrix and iteration number. To minimize the modified performance function, the penalty term should be zero. To minimize the extended performance function is equal to minimize the modified performance function. Then, the following equation is obtained.

$$\frac{\partial K}{\partial \theta} = 0.$$  \hspace{1cm} (41)

The control variable is renewed by the following equation:

$$\bar{\theta}^{(l+1)}(t) = \bar{\theta}^{(l)}(t) - \frac{1}{W^{(l)}} \text{grad}(J)^{(l)}.$$  \hspace{1cm} (42)

7 Numerical Study

7.1 Numerical Study 1

The numerical analysis of natural fluid flow is carried out using the three dimensional model. The finite element mesh and computational model are shown in Figs.3 and 4. The total number of nodes and elements are 14161 and 73728, respectively. On the bottom surface, temperature at points is specified to be equal to 1.0. On the top surface, temperature is set to be equal to 0.0. The boundary condition for velocity is non-slip condition on all the surface. The Rayleigh number and the Prandtl number are set to 1000.0 and 0.71, respectively. The time increment $\Delta t$ is 0.0003.

The natural fluid flow using the finite element method is carried out. The temperature of non-dimensional time $t=0.45$ is shown in Fig.7. The velocity of non-dimensional time $t=0.45$ is illustrated in Fig.8. It is seen that phenomenon of thermal fluid flow is simulated.
7.2 Numerical Study 2

The optimal control of temperature in natural fluid flow is performed. The finite element mesh and computation model are shown in Figs.5 and 6. The total number of nodes and elements are 2975 and 13056, respectively. The control and the specified boundaries are shown in Fig.6. The temperature on the specified boundary is shown in Fig.9. The boundary condition for velocity is given as non-slip condition on the all surface. In this study, the Rayleigh number and the Prandtl number are set to 1000.0 and 0.71, respectively. The time increment $\Delta t$ is 0.001. The object point is set as shown in Fig.6. Using the specified temperature shown in Fig.9, the natural fluid flow is occurred. The temperature at the object point shown in Fig.10 can be obtained. To reduce the temperature at the object point from initial to target temperature, the optimal control is carried out. The target temperature is set to 0.0. This optimal control problem is to find the control temperature on the control boundaries so as to minimize the performance function. The weighted gradient method is applied as the minimization technique.

The time history of the performance function is shown in Fig.11. The time history of the control temperature at the control boundary is shown in Fig.12. The time history of temperature at the object point is shown in Fig.13. At the target point, the temperature could be controlled to the objective temperature. The temperatures on $Y=0.5$ and $Z=0.125$ are shown in Fig.14. The temperature and the velocity without control at non-dimensional time $t=2.0$ are shown in Figs.15 and 17. The temperature and the velocity with control at non-dimensional time $t=2.0$ are shown in Figs.16 and 18. The temperature at the same altitude can be reduced by setting only one object point as shown in shown Fig.14. The optimal control of temperature in natural fluid flow can be carried out.

![Fig.5 : Finite Element Mesh](image1.png)  
![Fig.6 : Computational Model](image2.png)

8 Conclusion

The optimal control of temperature in the natural fluid flow is presented. The control temperature can be found so as to minimize the performance function using the weighted gradient method. It is shown that the temperature at object point can be reduced using the control temperature. The all temperature of the same altitude is reduced by setting only one object point. The temperature at the object point is well controlled using the present method.

References
Fig. 13: Temperature at Object Point

Fig. 14: Temperature Distribution (t=2.0)

Fig. 15: Temperature Without Control (t=1.6)

Fig. 16: Temperature With Control (t=1.6)
Fig. 17: Velocity Without Control
(t=1.6)

Fig. 18: Velocity With Control
(t=1.6)