Application of Second Order Adjoint Technique for Conduit Flow Problem

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SUMMARY

This paper presents the way to obtain the Newton gradient by using a traction given by the perturbation for the Lagrange multiplier. Conventionally, the second order adjoint model using the Hessian/Vector Products expressed by the product of the Hessian matrix and the perturbation of the design variables has been researched. However, in case that the boundary value would like to be obtained, this model can not be directly applied. Therefore, the conventional second order adjoint technique is extended to the boundary value determination problem and the second order adjoint technique is applied to the conduit flow problem in this paper. As the minimization technique, the Newton based method is employed. The BFGS method is applied to calculate the Hessian matrix which is used in the Newton based method and the a traction given by the perturbation for the Lagrange multiplier is used in the BFGS method.

KEY WORDS: Second Order Adjoint Technique; Newton Gradient; Traction of Perturbation for Lagrange Multiplier; BFGS Method; Newton Based Method; Finite Element Method

1. INTRODUCTION

In late years, the flood damage occurs frequently in an urban area. The purposes of sewage are the maintenance of water quality, the improvement of living environment and the exclusion of stormwater. A measure for stormwater is one of an important role of the sewer. Storm tends to occur frequently by abnormal weather or activation of a front by a typhoon. Therefore, to prevent the flood damage in an urban area, the countermeasure for stormwater is an urgent business. In late years, introduction of the flood control system is considered as one of the countermeasures for stormwater. In case that the flood control system is constructed, it is desired that many examinations will be carried out based on control theory and the flood control system will be adequately operated.
Thus, in case that the design variables are determined based on control theory, a performance function is defined as the performance index. This control problem can be expressed by the minimization problem of the performance function, and the problem is formulated under the constraint condition that can be expressed by the state equation. In generally, the gradient based method or the Newton based method is employed for minimizing the performance function. In this study, we focus on the Newton based method that is well known as the method by which the fast convergence rate can be obtained. It is necessary that the information of the Hessian matrix is obtained in case that the Newton based method is employed. In this study, the second order adjoint technique is applied to obtain the traction given by the perturbation for the Lagrange multiplier that has the information of exact Hessian matrix. The purpose of this study is to verify the effect of the second order adjoint technique by using the conduit flow problem. To compute the state equation and adjoint equation, the finite element method is employed. As the numerical experiments, to verify the propriety of the second order adjoint technique, a simple conduit flow problem is used. After that, this technique is applied to the flood control problem.

2. FORMULATIONS

2.1. First variation

To express the flow behavior in a conduit, the Saint-Venant equation is frequently employed. The Saint-Venant equation is written as,

\[ \dot{Q} + (QV)_x + gA (\cos \theta) h_x - g(s_0 A - fQ) = 0 \quad \text{in} \quad L \quad t \in [t_0, t_f], \quad (1) \]

\[ \dot{A} + Q_x = 0 \quad \text{in} \quad L \quad t \in [t_0, t_f], \quad (2) \]

where \( x \) is the longitudinal distance along the channel, \( t \) is the time, \( Q \) is the flow quantity, \( A \) is the cross section area, \( V \) is the water velocity, \( h \) is the surface level of the water in the conduit, \( \theta \) is the angle of the slope, \( g \) is the acceleration of gravity, \( s_0 \) is the slope of bottom of the conduit, and \( f \) is the coefficient of friction, and is given by \( (n^2 Q/R^{4/3} A) \). \( n \) is the Manning coefficient of roughness, \( R \) is the hydraulic mean depth, and is given by \( (A/S) \). \( S \) is the wetted perimeter. The diagram of the conduit flow is shown in Figure 1.

The initial and boundary conditions are written as,

\[
\begin{cases}
Q(t_0) = \hat{Q}(t_0) \quad \text{in} \quad L, \\
A(t_0) = \hat{A}(t_0) \quad \text{in} \quad L, \\
Q(t) = \hat{Q}(t) \quad \text{at} \quad X_1 \quad t \in [t_0, t_f], \\
A(t) = \hat{A}(t) \quad \text{at} \quad X_1 \quad t \in [t_0, t_f], \\
Q(t) = Q_{\text{Cont.}}(t) \quad \text{at} \quad X_{\text{Cont.}} \quad t \in [t_0, t_f],
\end{cases}
\quad (3)
\]

where \( X_1 \) means the points at which the boundary condition is given and \( X_{\text{Cont.}} \) mean the points at which the control discharge is given.
To apply the Galerkin method for the spatial discretization of the state equation, the following finite element equation can be obtained. The Crank-Nicolson scheme is applied to the temporal discretization for the finite element equation.

\[
\dot{M}Q + S(Q)_{x}V + S(V)_{x}Q + g(\cos\theta)S(A)_{x}h \\
- g(s_{0}MA - fMQ) = 0 \text{ in } L \quad t \in [t_0, t_f],
\]

(4)

\[
M\dot{A} + S_{x}Q = 0 \text{ in } L \quad t \in [t_0, t_f],
\]

(5)

where \( M, S_{x}, S_{x}(Q) \) and \( S_{h} \) mean the coefficient matrices of the finite element equation. In addition, the Preissmann slot model\(^{(10)} \) is included in the analysis of conduit flow.

In this section, the flood control problem is considered. To prevent the flood damage in an urban area, the stormwater tunnel reservoir is constructed in the underground. Here, let’s consider a system that the water flow in the conduit is adequately conducted to the stormwater tunnel reservoir by using the control device. The diagram of the flood control system is shown in Figure 2. To solve this problem, the performance function is defined as follows. The first term consists of physical value of the control target and can be denoted by the square form of the residual between the water level \( h \) and the target water level \( h_{\text{Target}} \). The second term means the evaluation term of the control discharge \( Q_{\text{Cont}} \) and can be expressed by the square form of the control discharge.

\[
J = \frac{1}{2} \int_{t_0}^{t_f} \int_{L} (h - h_{\text{Target}})^T P(h - h_{\text{Target}}) \, dx \, dt \\
+ \frac{1}{2} \int_{t_0}^{t_f} \int_{L} Q_{\text{Cont}}^T R Q_{\text{Cont}} \, dx \, dt,
\]

(6)
where $P$ and $R$ mean the weighting diagonal matrices. The performance function can be represented by the form of the cross section area $A$, because the water level $h$ is the function of cross section area $A$. Therefore, the performance function can be denoted as,

$$J = \frac{1}{2} \int_{t_0}^{t_f} \int_L (A - A_{Target})^T P (A - A_{Target}) \, dx \, dt$$

$$+ \frac{1}{2} \int_{t_0}^{t_f} \int_L Q_{Cont.}^T R Q_{Cont.} \, dx \, dt,$$

(7)

The purpose is to find the adequate control discharge $Q_{Cont.}$ to minimize the performance function $J$ under the constraints of the state equation, the initial condition and the boundary condition. The thing which the performance function is minimized means that the water level $h$ is as close as the target water level $h_{Target}$.

The performance function is constrained by the state equation, because the performance function can be expressed by the solution of the state equation. Therefore, the Lagrange multiplier method is introduced to the minimization problem with constraint condition. The performance function extended by the Lagrange multipliers and the state equation can be expressed as

$$J^* = J + \int_{t_0}^{t_f} \int_L \left\{ \lambda^T (M \dot{Q} + F(Q, A)) + \mu^T (M \dot{A} + G(Q)) \right\} \, dx \, dt,$$

(8)

where $\lambda$ and $\mu$ are the Lagrange multipliers for the flow quantity $Q$ and the cross section area $A$. $F(Q, A)$ and $G(Q)$ mean the steady terms for the momentum and the continuity equations. These functions are written as

$$F(Q, A) = S(Q)_{x,xx} V + S(V)_{,x} Q + g(cos\theta)S(A)_{,x} h$$

$$- g(s_0 MA - f MQ) \quad \text{in} \quad L \quad t \in [t_0, t_f],$$

(9)
APPLICATION OF SECOND ORDER ADJOINT TECHNIQUE

\[ G(Q) = S, \quad Q \in L, \quad t \in [t_0, t_f]. \]  

(10)

To derive the stationary condition, the first variation of the extended performance function is calculated. Consequently, the following first order adjoint equation can be obtained.

\[ -M^T \dot{\lambda} + \left( \frac{\partial F(Q, A)}{\partial Q} \right)^T \lambda + \left( \frac{\partial G(Q)}{\partial Q} \right)^T \mu = s, \quad L, \quad t \in [t_0, t_f], \]  

(11)

\[ -M^T \dot{\mu} + \left( \frac{\partial F(Q, A)}{\partial A} \right)^T \lambda + P^T (A - A_{\text{Target}}) = 0, \quad L, \quad t \in [t_0, t_f], \]  

(12)

In addition, the boundary and terminal conditions for the Lagrange multipliers can be obtained as

\[
\begin{aligned}
\lambda(t_f) &= \hat{\lambda}(t_f) \quad \text{in} \ L, \\
\mu(t_f) &= \hat{\mu}(t_f) \quad \text{in} \ L, \\
\lambda(t) &= 0 \quad \text{at} \ X_1, \quad t \in [t_0, t_f], \\
\mu(t) &= 0 \quad \text{at} \ X_1, \quad t \in [t_0, t_f], \\
\lambda(t) &= 0 \quad \text{at} \ X_{\text{Cont.}}, \quad t \in [t_0, t_f].
\end{aligned}
\]  

(13)

Moreover, the gradient of the extended performance function with respect to the control discharge can be obtained as

\[
\frac{\partial J^*}{\partial Q_{\text{Cont.}}} = s = -M^T \dot{\lambda}(t) + \left( \frac{\partial F(Q, A)}{\partial Q_{\text{Cont.}}} \right)^T \lambda(t) \\
+ \left( \frac{\partial G(Q)}{\partial Q_{\text{Cont.}}} \right)^T \mu(t) + R^T Q_{\text{Cont.}} \quad \text{at} \ X_{\text{Cont.}}, \quad t \in [t_0, t_f].
\]  

(14)

2.2. Derivation of perturbation equations for state variables and Lagrange multiplier

The following perturbation equation for the state variables can be derived by the formulation that the state equation is extended by the perturbation \( Q' \) and \( A' \) and the first order term consists of the perturbations \( Q' \) and \( A' \) is retained.

\[ M \dot{Q}' + \frac{\partial F(Q, A)}{\partial Q} Q' + \frac{\partial F(Q, A)}{\partial A} A' = 0, \quad L, \quad t \in [t_0, t_f], \]  

(15)

\[ M \dot{A}' + \frac{\partial G(Q)}{\partial Q} Q' = 0, \quad L, \quad t \in [t_0, t_f], \]  

(16)

Here, the initial and boundary conditions for the perturbation equation can be denoted as

\[
\begin{aligned}
Q'(t_0) &= \hat{Q}'(t_0) \quad \text{in} \ L, \\
A'(t_0) &= \hat{A}'(t_0) \quad \text{in} \ L, \\
Q'(t) &= \hat{Q}'(t) \quad \text{at} \ X_1, \quad t \in [t_0, t_f], \\
A'(t) &= \hat{A}'(t) \quad \text{at} \ X_1, \quad t \in [t_0, t_f], \\
Q'(t) &= Q'_{\text{Cont.}}(t) \quad \text{at} \ X_{\text{Cont.}}, \quad t \in [t_0, t_f].
\end{aligned}
\]  

(17)
Similarly, the following second order adjoint equation can be obtained by the formulation that the first order adjoint equation is extended by the perturbations $\lambda'$ and $\mu'$.

\[ -M^T \dot{\lambda}' + \left( \frac{\partial^2 F(Q, A)}{\partial Q^2} Q' \right)^T \lambda + \left( \frac{\partial^2 G(Q)}{\partial Q^2} Q' \right)^T \mu' \]

\[ + \left( \frac{\partial F(Q, A)}{\partial Q} \right)^T \lambda' + \left( \frac{\partial G(Q)}{\partial Q} \right)^T \mu' \]

\[ = \left( \frac{\partial J^*}{\partial Q^2_{\text{Cont.}}} \right) Q_{\text{Cont.}}(= s') \quad \text{in} \quad L \quad t \in [t_0, t_f], \quad \tag{18} \]

\[ -M^T \dot{\mu}' + \left( \frac{\partial^2 F(Q, A)}{\partial A^2} A' \right)^T \lambda' + \left( \frac{\partial F(Q, A)}{\partial A} \right)^T \lambda' + P^T A' = 0 \quad \text{in} \quad L \quad t \in [t_0, t_f], \tag{19} \]

The above equation can be formulated based on the assumption that the following conditions are satisfied. The following conditions are consists of the terms expressed by the products of the perturbations.

\[ \left( \frac{1}{2} Q^T \frac{\partial^3 F(Q, A)}{\partial Q^3} Q' \right)^T \lambda + \left( \frac{\partial^2 F(Q, A)}{\partial Q^2} \right)^T \lambda' + \left( \frac{\partial^2 F(Q, A)}{\partial A^2} A' + \frac{1}{2} A^T \frac{\partial^3 F(Q, A)}{\partial A^2} A' \right)^T \lambda' = 0 \quad \text{in} \quad L \quad t \in [t_0, t_f]. \quad \tag{20} \]

Here, the terminal and boundary conditions for the second order adjoint equation can be written as

\[
\begin{align*}
\lambda(t_f) &= \tilde{X}(t_f) \quad \text{in} \quad L, \\
\mu'(t_f) &= \tilde{\mu}'(t_f) \quad \text{in} \quad L, \\
\lambda(t) &= 0 \quad \text{at} \quad X_1, \quad t \in [t_0, t_f], \\
\mu'(t) &= 0 \quad \text{at} \quad X_1, \quad t \in [t_0, t_f], \\
\lambda(t) &= 0 \quad \text{at} \quad X_{\text{Cont.}}, \quad t \in [t_0, t_f]. \tag{22} \\
\end{align*}
\]

Finally, the traction given by the perturbation for the Lagrange multiplier can be calculated as

\[ \left( \frac{\partial J^*}{\partial Q^2_{\text{Cont.}}} \right) Q_{\text{Cont.}} = s' = \]

\[ -M^T \dot{\lambda}' + \left( \frac{\partial^2 F(Q, A)}{\partial Q^2_{\text{Cont.}}} Q_{\text{Cont.}} \right)^T \lambda + \left( \frac{\partial^2 G(Q)}{\partial Q^2_{\text{Cont.}}} \right)^T \mu' \]

\[ + \left( \frac{\partial F(Q, A)}{\partial Q_{\text{Cont.}}} \right)^T \lambda' + \left( \frac{\partial G(Q)}{\partial Q_{\text{Cont.}}} \right)^T \mu' \]

\[ + R^T Q'_{\text{Cont.}} \quad \text{at} \quad X_{\text{Cont.}} \quad t \in [t_0, t_f]. \quad \tag{23} \]
3. MINIMIZATION TECHNIQUE

3.1. Derivation of Newton equation

In case that $J(Q_{\text{Cont.}})$, $\frac{\partial J}{\partial Q_{\text{Cont.}}} (Q_{\text{Cont.}})$, and $\frac{\partial^2 J}{\partial Q^2_{\text{Cont.}}} (Q_{\text{Cont.}})$ can be obtained, $J(Q_{\text{Cont.}})$ can be approximated by Taylor expansion as

$$J(Q_{\text{Cont.}}) = J(Q_{\text{Cont.}}) + \frac{\partial J(Q_{\text{Cont.}})}{\partial Q_{\text{Cont.}}} (Q_{\text{Cont.}} - Q_{\text{Cont.}}) + (Q_{\text{Cont.}} - Q_{\text{Cont.}}) \frac{\partial^2 J(Q_{\text{Cont.}})}{\partial Q^2_{\text{Cont.}}} (Q_{\text{Cont.}} - Q_{\text{Cont.}})$$

(24)

It is assumed that the global minimum point for right hand side terms of the above equation is $Q_{\text{Cont.}}^{(l+1)}$. Therefore, the following equation can be obtained by the stationary condition;

$$\frac{\partial J(Q_{\text{Cont.}})}{\partial Q_{\text{Cont.}}} + \frac{\partial^2 J(Q_{\text{Cont.}})}{\partial Q^2_{\text{Cont.}}} (Q_{\text{Cont.}}^{(l+1)} - Q_{\text{Cont.}}^{(l)}) = 0,$$

(25)

where $\frac{\partial J(Q_{\text{Cont.}})}{\partial Q_{\text{Cont.}}}$ and $\frac{\partial^2 J(Q_{\text{Cont.}})}{\partial Q^2_{\text{Cont.}}}$ are set as

$$\frac{\partial J(Q_{\text{Cont.}})}{\partial Q_{\text{Cont.}}} = g^{(l)},$$

(26)

$$\frac{\partial^2 J(Q_{\text{Cont.}})}{\partial Q^2_{\text{Cont.}}} = H^{(l)}.$$

(27)

Hence, the equation of stationary condition can be expressed as

$$g^{(l)} + H^{(l)} (Q_{\text{Cont.}}^{(l+1)} - Q_{\text{Cont.}}^{(l)}) = 0,$$

(28)

where $g^{(l)}$ and $H^{(l)}$ mean the steepest decent direction and Hessian matrix. The above equation can be formulated as

$$Q_{\text{Cont.}}^{(l+1)} = Q_{\text{Cont.}}^{(l)} - H^{(l)-1} g^{(l)}.$$

(29)

Finally, the renewal equation of control value can be obtained as follows.

$$Q_{\text{Cont.}}^{(l+1)} = Q_{\text{Cont.}}^{(l)} + d^{(l)},$$

(30)

where $d^{(l)}$ means the Newton direction and this direction can be calculated as

$$-H^{(l)-1} g^{(l)} = d^{(l)}.$$

(31)
The equation which can be obtained the Newton direction can be transformed as
\[ H(l) d(l) = -g(l), \]  
(32)
where this equation is called the Newton equation.
In addition, the control value is updated by the step length \( \alpha(l) \) and the Newton direction \( d(l) \) in the optimization problem. The renewal equation which is used in the optimization problem is expressed as
\[ Q^{(l+1)}_{\text{Cont.}} = Q^{(l)}_{\text{Cont.}} + \alpha^{(l)} d^{(l)}, \]  
(33)
where the step length \( \alpha^{(l)} \) is obtained by the line search algorithm. The line search algorithm is a method that the adequate step length \( \alpha^{(l)}(h) \) is obtained by gradually changing the step increment \( h \).

3.2. Solver of Newton equation

The Newton equation can be represented as
\[ H(l) d(l) = -g(l), \]  
(34)
The conjugate gradient method is available for the solver of the Newton equation, because the Hessian matrix \( H(l) \) is a symmetric matrix. The computational algorithm of the conjugate gradient method is shown as follows.

1. Assume initial Newton direction \( d_0 \), set allowable constants \( \epsilon \) and \( l = 0 \).
   Set \( r_0 \) and \( p_0 \); \( r_0 = -g(l) - H(l) d_0, p_0 = r_0 \).
2. Determine \( \alpha; \alpha = \frac{(r^{(l)}, r^{(l)})}{\langle p^{(l)}, H^{(l)} p^{(l)} \rangle} \)
3. Compute the Newton direction \( d^{(l+1)}; d^{(l+1)} = d^{(l)} + \alpha^{(l)} p^{(l)} \)
4. Compute the residual \( r^{(l+1)}; r^{(l+1)} = r^{(l)} - \alpha^{(l)} H^{(l)} p^{(l)} \)
5. Determine \( \beta; \beta = \frac{\langle r^{(l+1)}, r^{(l+1)} \rangle}{\langle r^{(l)}, r^{(l)} \rangle} \)
6. Compute the gradient \( p^{(l+1)}; p^{(l+1)} = r^{(l)} + \beta^{(l)} p^{(l)} \)
7. Check the convergence; if \( \|d^{(l+1)} - d^{(l)}\| < \epsilon \) then stop, else go to step 2.

Excepting the above method, the method using Lanczos tridiagonalization can be also applied to obtain the Newton direction. \(^{1-11}\)

3.3. BFGS method

The DFP method (Davidon-Fletcher-Powell method) \(^{14}\) and the BFGS method (Broyden-Fletcher-Goldfarb-Shanno method) \(^{15}\) which are formulated by the secant equation \(^{16}\) are enumerated as the methodology which is used to obtain a Hessian matrix. These methods are ensured that the Hessian matrix is positive definite and symmetric. Therefore, these methods are frequently used to calculate the Hessian matrix.

In this study, the BFGS method is applied to obtain the Hessian matrix. The update equation of Hessian matrix by using the BFGS method can be written as
APPLICATION OF SECOND ORDER ADJOINT TECHNIQUE

\[ H^{(l+1)} = H^{(l)} + \frac{g^{k^T} Q_{\text{Cont.}}^{k^T}}{Q_{\text{Cont.}}^{k^T} Q_{\text{Cont.}}^{k^T} H^{(l)}}, \]

where \( Q_{\text{Cont.}}, g' \) and \( s' \) are denoted as

\[
\begin{align*}
H^{(l)} &= \frac{\partial^2 J^*}{\partial Q^2_{\text{Cont.}}} \\
Q_{\text{Cont.}}^{l^T} &= Q_{\text{Cont.}}^{(l+1)} - Q_{\text{Cont.}}^{(l)} \\
g^{k^T} &= \frac{\partial J^*(l+1)}{\partial Q_{\text{Cont.}}} - \frac{\partial J^*(l)}{\partial Q_{\text{Cont.}}} = g^{(l+1)} - g^{(l)}.
\end{align*}
\]

Here, \( s' \) which means the traction given by the perturbation for the Lagrange multiplier can be shown as

\[ s^{(l)^T} = H^{(l)} Q_{\text{Cont.}}^{k^T} = \frac{\partial^2 J^*}{\partial Q_{\text{Cont.}}^2} Q_{\text{Cont.}}^{k^T}. \]

Consequently, the update equation of Hessian matrix can be represented as

\[ H^{(l+1)} = H^{(l)} + \frac{g^{k^T} g^{k^T} Q_{\text{Cont.}}^{k^T}}{Q_{\text{Cont.}}^{k^T} Q_{\text{Cont.}}^{k^T} Q_{\text{Cont.}}}, \]

In generally, the product between the Hessian matrix \( H \) and the perturbation \( Q_{\text{Cont.}}^l \) can not be obtained directly. Conventionally, this product is calculated by using an approximated Hessian matrix.

On the other hand, the traction given by the perturbation for the Lagrange multiplier expressed by exact Hessian matrix can be obtained in this approach. Therefore, it is considered that the more exact Newton direction can be obtained in comparison with conventional method.\(^\text{17}\)

3.4. Newton based method

The computational algorithm of the Newton based method is similar to that of the gradient methods which mean the Fletcher-Reeves method, the Sakawa-Shindo method, the Conjugate Direction method, the Steepest Decent method and so on. The difference between the Newton based method and the gradient based methods is the decent direction which is used for the renewal equation of the control variables. In the gradient methods, the steepest decent direction is directly used for the renewal equation of control variables. On the other hand, in the Newton based method, the Newton direction which is obtained by the steepest decent direction is used for the renewal equation of control variables.

The computational algorithm is shown below.

1. Choose an initial control value \( Q_{\text{Cont.}}^{(l)} \), an initial Hessian Matrix \( H^{(l)} \) and a convergence criteria \( \epsilon \).
2. Compute state value \( (Q^{(l)}, A^{(l)}) \) by the state equation.
3. Compute the performance function \( J^{(l)} \).
4. Compute the Lagrange Multiplier \( (\lambda^{(l)}, \mu^{(l)}) \) and the steepest decent direction \( g^{(l)} \).
5. Solve the Newton Equation \( H^{(l)}d^{(l)} = -g^{(l)} \) by the conjugate gradient method.
6. Generate a new control value \( Q_{\text{Cont.}}^{(l+1)} \) by the Newton direction \( d^{(l+1)} \) and the step length \( \alpha^{(l+1)} \).
7. Compute state value \( (Q^{(l+1)}, A^{(l+1)}) \) by the state equation.
8. Compute the performance function \( J^{(l+1)} \).
9. Compute the Lagrange Multiplier \( (\lambda^{(l+1)}, \mu^{(l+1)}) \) and the steepest decent direction \( g^{(l+1)} \).
10. Check the convergence; if \( \|Q_{\text{Cont.}}^{(l+1)} - Q_{\text{Cont.}}^{(l)}\| < \epsilon \) then stop, else go to step 11.
11. Compute the perturbed solution \( (Q'^{(l+1)}{\text{Cont.}}, A'^{(l+1)}) \) by the perturbation equation for state variables.
12. Compute the Lagrange Multiplier \( (\lambda'^{(l+1)}, \mu'^{(l+1)}) \) and the traction given by the perturbation for the Lagrange multiplier \( s'^{(l+1)} \).
13. Compute the Hessian matrix \( H^{(l+1)} \) by the \( (Q_{\text{Cont.}}^{(l+1)} - Q_{\text{Cont.}}^{(l)}), (g^{(l+1)} - g^{(l)}) \) and \( s'^{(l)} \).
14. Solve the Newton Equation \( H^{(l+1)}d^{(l+1)} = -g^{(l+1)} \) by the conjugate gradient method and go to 6.

The initial Hessian matrix is generally set an unit matrix.
4. NUMERICAL EXPERIMENTS

4.1. Verification of computational algorithm by second order adjoint technique

A simple conduit flow problem is used as a numerical study. The diagram of this study is illustrated in Figure 3. The length of conduit is 20m and the observation point is set at the point A. The total points of node and element are respectively 200 and 199. The slope of conduit is set 0.5%.

![Figure 3. Computational model and finite element division](image)

The computational conditions for this analysis are listed in Table 1. The weighting constant $P$ at the target point is 1.0 and at other points is 0.0. The weighting constant $R$ at the control point is 0.0 and at other points are also 0.0.

<table>
<thead>
<tr>
<th>Table 1 Computational conditions</th>
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<tbody>
<tr>
<td>Number of time steps</td>
</tr>
<tr>
<td>Time increment $\Delta t$ (sec)</td>
</tr>
<tr>
<td>Manning coefficient of roughness $n$ (sec/m$^{1/3}$)</td>
</tr>
</tbody>
</table>

In case that the flow shown in Figure 4 is given from the upstream side of conduit, the time history of water level shown in Figure 5 can be obtained as the computational result. Here, let’s consider the problem that we have the time history of water level shown in Figure 5 and would like to know the time history of inflow discharge. The purpose of this study is to find the inflow discharge such that the water level approaches to the observed water level.
Figure 4. Time history of inflow discharge

Figure 5. Time history of water level at observation point
4.1.1. **Computational results by first order adjoint technique**  
In this section, it discusses the problem that the gradient method is applied to this study. The computational results are shown in Figures 6 to 8. Figure 6 shows the convergence history of performance function. The performance function gradually decreases and finally converges. Consequently, the water level at the observation point approaches to the observed water level as shown in Figure 7, and the inflow discharge can be obtained as shown in Figure 8. It is found that the computed inflow discharge is good agreement with the assumed inflow discharge.

![Figure 6. Convergence history of performance function](image-url)
Figure 7. Comparison between computed water level and assumed water level

Figure 8. Comparison between computed inflow discharge and assumed inflow discharge
4.1.2. **Computational results by second order adjoint technique**

In this section, it discusses the problem that the Newton based method by the BFGS method using the traction given by the perturbation for the Lagrange multiplier is applied to this study. The computational results are shown in Figures 9 to 11. Figure 9 shows the convergence history of performance function. The performance function gradually decreases and finally converges. The first convergence rate can be obtained in comparison with the case the gradient method is applied. Consequently, the water level at the observation point approaches to the observed water level as shown in Figure 10, and the inflow discharge can be obtained as shown in Figure 11. It is found that the computed inflow discharge is good agreement with the assumed inflow discharge as well as the result of the gradient method. In addition, the comparison of computational time between the first order adjoint technique and the second order adjoint technique is shown in Table 2. The computational time could be reduced by 28\% in comparison with that of the first order adjoint technique.

<table>
<thead>
<tr>
<th>Table 2 Comparison of computational time</th>
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<tr>
<td>First order adjoint technique</td>
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<tr>
<td>Second order adjoint technique</td>
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</table>

Figure 9. Convergence history of performance function
Figure 10. Comparison between computed water level and assumed water level

Figure 11. Comparison between computed inflow discharge and assumed inflow discharge
4.2. Application to flood control problem

In this section, the second order adjoint technique is applied to a flood control problem. Here, an urban area shown in Figure 12 is used as the example of practical problem. Figure 12 shows the aerophotograph in the examination area and the flood expectation district. It is said that the flood will be occurred at the area shown in the photograph in case of heavy rainfall that the rainfall intensity is 50 mm/hr. Therefore, a countermeasure that the stormwater tunnel reservoir will be constructed in the underground is considered by the local authority. Thus, the purpose of this study is to find the adequate control discharge to the stormwater reservoir so as to prevent the flood damage. Figures 13 and 14 show the boundary line for the each catchment districts in the examination area and the diagram of conduits network. To compute the flow behavior, the conduits network is divided into 2,565 elements. The total points of nodes are 2,566. The inflow boundary conditions which mean the rainfall conditions are given at the starting points of conduits in each catchment districts. The inflow boundary condition is frequently calculated by the rational equation shown in eq.(39).

\[ Q_{\text{inflow}} = \frac{1}{360} CIA \]  

Here, \( Q_{\text{inflow}} \), \( C \), \( I \) and \( A \) mean the inflow discharge, the runoff coefficient, the rainfall intensity and the catchment area. The areas for each catchment districts are shown in Table 3 and the time history of rainfall intensity is shown in Figure 15. This time history of rainfall intensity means the rainfall of 50 mm/hr. In addition, the runoff coefficient is set 0.8 and the other computational conditions are shown in Table 4. In addition, the weighting constant \( P \) at the target points is 1.0 and at other points is 0.0. The weighting constant \( R \) at the control points is 0.0 and at other points are also 0.0. In case that the weighting constant \( R \) is set 0.0 at the control points, that means the scale of control device can be ignored.
Figure 12. Flood expectation district

Figure 13. Boundary lines for each chatchment districts
Figure 14. Diagram of conduits network

Figure 15. Time history of rainfall intensity
Table 3 Areas for each catchment districts

<table>
<thead>
<tr>
<th>Number</th>
<th>Area (ha)</th>
<th>Number</th>
<th>Area (ha)</th>
<th>Number</th>
<th>Area (ha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.1</td>
<td>6.69</td>
<td>No.9</td>
<td>3.99</td>
<td>No.17</td>
<td>0.59</td>
</tr>
<tr>
<td>No.2</td>
<td>1.76</td>
<td>No.10</td>
<td>1.70</td>
<td>No.18</td>
<td>4.23</td>
</tr>
<tr>
<td>No.3</td>
<td>2.02</td>
<td>No.11</td>
<td>1.98</td>
<td>No.19</td>
<td>1.85</td>
</tr>
<tr>
<td>No.4</td>
<td>3.52</td>
<td>No.12</td>
<td>2.42</td>
<td>No.20</td>
<td>1.69</td>
</tr>
<tr>
<td>No.5</td>
<td>3.44</td>
<td>No.13</td>
<td>3.40</td>
<td>No.21</td>
<td>1.48</td>
</tr>
<tr>
<td>No.6</td>
<td>1.20</td>
<td>No.14</td>
<td>3.76</td>
<td>No.22</td>
<td>1.77</td>
</tr>
<tr>
<td>No.7</td>
<td>5.05</td>
<td>No.15</td>
<td>2.94</td>
<td>——</td>
<td>——</td>
</tr>
<tr>
<td>No.8</td>
<td>3.80</td>
<td>No.16</td>
<td>1.24</td>
<td>Total</td>
<td>60.50</td>
</tr>
</tbody>
</table>

Table 4 Computational conditions

<table>
<thead>
<tr>
<th>Number of time steps</th>
<th>3,600</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time increment $\Delta t$ (sec)</td>
<td>0.50</td>
</tr>
<tr>
<td>Manning coefficient of roughness $n$ (sec/m$^{1/3}$)</td>
<td>0.013</td>
</tr>
<tr>
<td>Runoff coefficient $C$</td>
<td>0.80</td>
</tr>
</tbody>
</table>

In the flood expectation district, the target points are set two points in the main conduit. Additionally, the control points are set two points over the stormwater tunnel reservoir. The location of the target and control points is shown in Figure 16.

Figure 16. Location of target and control points

The purpose of this study is to obtain the adequate control discharge at the control points such that the water level at the target points reduce to the target water level. The target water level is set as the initial water level 1.0m.
As the computational results, the variation of performance function is shown in Figure 17. The value of the performance function monotonously decreases and finally converges. Consequently, the adequate control discharge shown in Figures 18 and 19 could be obtained. Figure 18 means the time history of the control discharge at the control point No.1, and Figure 19 means the time history of the control discharge at the control point No.2. In addition, the time history of water level at the target points No.1 and No.2 are shown in Figures 20 and 21. The solid line means the water level with control case and the broken line means the water level without control case. At the target points No.1 and No.2, the water level could be reduced to the target water level by giving the adequate control discharge.

Figure 17. Convergence history of performance function
Figure 18. Time history of control discharge at control point No.1

Figure 19. Time history of control discharge at control point No.2
Figure 20. Time history of water level at target point No.1

Figure 21. Time history of water level at target point No.2
5. CONCLUSIONS

In this paper, the Newton based method by the BFGS method using the traction given by the perturbation for the Lagrange multiplier was applied to the conventional BFGS method. As state equations, the Saint-Venant equation was employed to express water behavior. The Galerkin method using the bubble function element and the Crank-Nicolson method, respectively, were used for the spatial and temporal discretizations. The Newton based method by the BFGS method using the traction given the perturbation for the Lagrange multiplier were used as the minimization technique. As the numerical examples, the estimation problem of inflow discharge and the flood control problem were used. The conclusions of this study are written as follows.

- The propriety of the second order adjoint technique for the conduit flow problem could be confirmed.
- In case that the Newton based method was employed, the fast convergence rate could be confirmed.
- In the verification problem, the computational time could be reduced by 28% in comparison with that of the first order adjoint technique.
- In the flood control problem, the adequate control discharge could be obtained by the second order adjoint technique.

For the future, it is desired that this methodology is applied to the other boundary value determination problems.
REFERENCES


