Optimal Control of Dissolved Oxygen in Shallow Water Flow

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Abstract

The purpose of this paper is to present a control method of dissolved oxygen (DO) using the first order adjoint method. In the optimal control theory, the control variable is obtained by the minimization of the performance function. The gradient of performance function updates the control variable. The control variable is computed so as to minimize the performance function under the constraint of the state equations and boundary conditions. In this research, velocity of river is defined as control variable. As the state equation, the coupled the shallow water equation with the advection diffusion equation is applied. DO which is diffused by the shallow water flow is controlled by the input water flow. In the numerical study, example of optimal control of DO is shown using simple rectangular model and Teganuma river.

Key words : Finite Element Method, First Order Adjoint Equation, Water Purification Control, Couple of State Equation

1 Introduction

In recent years, the water state of long rivers become recovered with technical improvement of the water maintenance in Japan. However, this is hardly seen in small rivers. Water pollution problem assumes serious proportions. In this research, dissolved oxygen (DO) is picked up as one of the indices of water state. As practical phenomena, the Teganuma river is analyzed. The Teganuma river is located in Chiba prefecture in Japan. This river is picked up as a practical phenomenon because of serious pollution problem. Recently, the water conducting project is performed in part of upstream of this river. However, water does not flow because water velocity is too small. The optimal velocity is needed to diffuse clean water for this river. If the optimal velocity is found, the higher DO can be diffusing in this river. Then river can be cleaned up. Therefore, optimal control of DO is carried out to find the optimal velocity. In this study, water velocity is calculated as the control variable.

The control variable is computed so as to minimize the performance function under the constraint of the state equation and boundary conditions. The performance function is defined the square sum of difference between the computed and the target DO. The minimization of the performance function means that computed value becomes as close as possible to the target value. The first order adjoint equation can be derived by the state equation and the boundary conditions. The first order adjoint equation is solved by the Lagrange multiplier method. The Lagrange multiplier method is suitable for the minimization problem with constraint condition. As the minimization technique, the weighted gradient method is applied. It has been used widely for the minimization technique. Control variable is updated by the gradient which is obtained by the first order adjoint equation.
As the state equation, the coupled the shallow water equation with the advection diffusion equation is applied. As the discretization technique, the Crank-Nicolson method in the temporal direction and the the finite element method using stabilized bubble function in the space direction are applied. The indicial notation and summation convention with repeated indices are used.

2 State Equation

The non-linear shallow water equation is used as the state equation. The non-linear shallow water equation can be written as follows:

\[
\begin{align*}
    \dot{u}_i + u_j u_{i,j} + g(\xi + \eta + h)_{,i} - \nu(u_{i,j} + u_{j,i})_{,j} + f u_i &= 0, \\
    \dot{\xi} + \{(\xi + h)u_i\}_{,i} &= 0,
\end{align*}
\]

where \(u, g, \xi, \eta\) and \(h\) are water velocity, gravitational acceleration, water elevation, bed elevation and water depth, respectively. The coefficient of kinematic eddy viscosity \(\nu\) and the bottom friction \(f\) are expressed as follows:

\[
\nu = \frac{k_l}{6} u_* (\xi + h), \quad f = \frac{u_*}{(\xi + h)}.
\]

The boundary conditions are given as follows:

\[
\begin{align*}
    u_i(t) &= \hat{u}_i \quad \text{on} \quad \Gamma_D, \\
    \xi(t) &= \hat{\xi} \quad \text{on} \quad \Gamma_D, \\
    u_i(t) &= u_i n_i = \hat{u}_i \quad \text{on} \quad \Gamma_N, \\
    u_i(t) &= U_i \quad \text{on} \quad \Gamma_C,
\end{align*}
\]

where the boundaries \(\Gamma_D, \Gamma_N\) and \(\Gamma_C\) are the Dirichlet, Neumann and control boundary, respectively and \(U_i\) is control variable.

The initial conditions are given as follows:

\[
\begin{align*}
    u_i(t_0) &= \hat{u}_i \quad \text{in} \quad \Omega, \\
    \xi(t_0) &= \hat{\xi} \quad \text{in} \quad \Omega,
\end{align*}
\]

The advection diffusion equation can be written as follows:

\[
\dot{c} + (u_i c)_{,i} - \kappa c_{,ii} = 0,
\]
where \( c \) and \( \kappa \) are concentration of dissolved oxygen and the diffusion coefficient, respectively.

The boundary conditions are given as follows:

\[
c(t) = \hat{c} \quad \text{on} \quad \Gamma_D,
\]

\[
b(t) = \kappa c_i n_i = \hat{b} \quad \text{on} \quad \Gamma_N,
\]

\[
c(t) = \hat{c} \quad \text{on} \quad \Gamma_C,
\]

The initial conditions are given as follows:

\[
u_i(t_0) = \hat{u}_i \quad \text{in} \quad \Omega,
\]

\[
c(t_0) = \hat{c} \quad \text{in} \quad \Omega.
\]

3 Discretization Technique

3.1 Temporal Discretization

As for the temporal discretization of the basic equations, the Crank-Nicolson method is applied:

\[
\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_j u_i n_{i,j} + g(\xi^{n+\frac{1}{2}} + \eta + h)_{,i} - \nu(u_i n_{i,j} + u_j n_{j,i})_{,j} + f u_i n_{i} = 0,
\]

\[
\frac{\xi^{n+1} - \xi^n}{\Delta t} + u_i n_{,i} + \xi u_i n_{,i} = 0,
\]

\[
\frac{c^{n+1} - c^n}{\Delta t} + u_i c_{,i} + c u_i n_{,i} + \kappa c_{,i} = 0,
\]

where,

\[
u_i^{n+1} = \frac{1}{2}(u_i^{n+1} + u_i^n), \quad \xi^{n+1} = \frac{1}{2}(\xi^{n+1} + \xi^n) \quad c^{n+1} = \frac{1}{2}(c^{n+1} + c^n).
\]

3.2 Spatial Discretization

3.2.1 Bubble Function Interpolation

As for the spatial discretization of the basic equations, the finite element method based on the bubble function interpolation is applied:

\[
u_i = \Phi_1 u_{i1} + \Phi_2 u_{i2} + \Phi_3 u_{i3} + \Phi_4 \tilde{u}_{i4},
\]

\[
\tilde{u}_{i4} = u_{i4} - \frac{1}{3}(u_{i1} + u_{i2} + u_{i3}),
\]

\[
\xi = \Phi_1 \xi_1 + \Phi_2 \xi_2 + \Phi_3 \xi_3 + \Phi_4 \tilde{\xi}_4,
\]

\[
\tilde{\xi}_4 = \xi_4 - \frac{1}{3}(\xi_1 + \xi_2 + \xi_3),
\]

\[
c = \Phi_1 c_1 + \Phi_2 c_2 + \Phi_3 c_3 + \Phi_4 \tilde{c}_4,
\]

\[
\tilde{c}_4 = c_4 - \frac{1}{3}(c_1 + c_2 + c_3),
\]

\[
\Phi_1 = L_1, \quad \Phi_2 = L_2, \quad \Phi_3 = L_3, \quad \Phi_4 = 27L_1 L_2 L_3.
\]
3.2.2 Stabilized Form

In the bubble function, the numerical stabilization is not enough. Therefore the stabilized parameter is used for stabilization. The stabilized parameter of the shallow water equation and the advection diffusion equation $\tau_{eB_u}$, $\tau_{eB_t}$ and $\tau_{eB_C}$ can be written as follows:

for the momentum equation of shallow water equation,

$$\tau_{eB_u} = \frac{\langle \phi_e, 1 \rangle^2_{\Omega_e} A_e^{-1}}{\Delta t||\phi_e||^2_{\Omega_e} + \frac{1}{2}[\nu + \bar{\nu}]||\phi_e||^2_{\Omega_e} + f||\phi_e||^2_{\Omega_e}}.$$  \hspace{1cm} (27)

for the continuity equation,

$$\tau_{eB_t} = \frac{\langle \phi_e, 1 \rangle^2_{\Omega_e} A_e^{-1}}{\Delta t||\phi_e||^2_{\Omega_e} + \frac{1}{2}[\kappa + \bar{\nu}]||\phi_e||^2_{\Omega_e}}.$$  \hspace{1cm} (28)

and for the advection diffusion equation,

$$\tau_{eB_C} = \frac{\langle \phi_e, 1 \rangle^2_{\Omega_e} A_e^{-1}}{\Delta t||\phi_e||^2_{\Omega_e} + \frac{1}{2}[\kappa + \bar{\nu}]||\phi_e||^2_{\Omega_e} + \frac{1}{2}[\nu]|\phi_e||^2_{\Omega_e} + f||\phi_e||^2_{\Omega_e}}.$$  \hspace{1cm} (29)

where, $\bar{\nu}$ is stabilizing parameter. $\tau_{eB_u}$, $\tau_{eB_t}$ and $\tau_{eB_C}$ are determined to be equivalent to $\tau_{eS}$ using the stabilized finite element method:

for the momentum equation of shallow water equation,

$$\tau_{eB_u} = \left(\frac{1}{2} \tau_{eS_u} + \frac{\alpha}{\Delta t}\right)^{-1}, \quad \tau_{eS_u}^{-1} = \left[\left(\frac{2|U_i|}{h_e}\right)^2 + \left(\frac{4\nu}{h_e^2}\right)^2 + \left(\frac{u_+}{\xi + h}\right)^2\right]^{\frac{1}{2}}.$$  \hspace{1cm} (30)

for the continuity equation,

$$\tau_{eB_t} = \left(\frac{1}{2} \tau_{eS_t} + \frac{\alpha}{\Delta t}\right)^{-1}, \quad \tau_{eS_t}^{-1} = \left[\left(\frac{2|U_i|}{h_e}\right)^2 + \left(\frac{4\nu}{h_e^2}\right)^2 + \left(\frac{u_+}{\xi + h}\right)^2\right]^{\frac{1}{2}}.$$  \hspace{1cm} (31)

and for the advection diffusion equation,

$$\tau_{eB_C} = \left(\frac{1}{2} \tau_{eS_C} + \frac{\alpha}{\Delta t}\right)^{-1}, \quad \tau_{eS_C}^{-1} = \left[\left(\frac{2|U_i|}{h_e}\right)^2 + \left(\frac{4\kappa}{h_e^2}\right)^2\right]^{\frac{1}{2}}.$$  \hspace{1cm} (32)

where,

$$\alpha = \frac{A_e||\phi_e||^2_{\Omega_e}}{\langle \phi_e, 1 \rangle^2_{\Omega_e}} \quad h_e = \sqrt{2A_e}, \quad |U_i| = \sqrt{u^2 + v^2 + g(\xi + h)}, \quad |U_C| = \sqrt{u^2 + v^2}.$$  \hspace{1cm} (33)

Therefore, the stabilized parameter for the momentum and the continuity equation of the shallow water equation and for the advection diffusion equation can be written as follows:
for the momentum equation of shallow water equation,

\[(\nu + \tilde{\nu})ii|\Omega_e|^2 = \frac{\langle \phi_e, 1 \rangle^2}{A_e} r^{-1}_{es} - f \|\phi_e\|_{\Omega_e}^2, \quad (34)\]

for the continuity equation,

\[\tilde{\nu}jj|\phi_e,j\|^2_{\Omega_e} = \frac{\langle \phi_e, 1 \rangle^2}{A_e} r^{-1}_{es}, \quad (35)\]

and for the advection diffusion equation,

\[(\kappa + \tilde{\nu})|\phi_e,j\|^2_{\Omega_e} = \frac{\langle \phi_e, 1 \rangle^2}{A_e} r^{-1}_{es}, \quad (36)\]

these can be effected to barycenter point of viscosity term. \(\Omega_e\) is element domain, \((a, b)_{\Omega_e}, \|\phi_e\|^2_{\Omega_e}\) and \(A_e\) is expressed as follows:

\[(a, b)_{\Omega_e} = \int_{\Omega_e} ab d\Omega, \quad \|\phi_e\|^2_{\Omega_e} = \langle \phi_e, \phi_e \rangle_{\Omega_e}, \quad A_e = \int_{\Omega_e} d\Omega, \quad (37)\]

the integration of bubble function is expressed as follows:

\[\langle \phi_e, 1 \rangle_{\phi_e} = \frac{A_e}{3}, \quad \|\phi_e\|^2_{\Omega_e} = \frac{A_e}{6}. \quad (38)\]

### 4 Optimal Control

#### 4.1 Performance Function

The control variable is computed so as to minimize the performance function under the constraint of the state equation and boundary conditions. The minimization of the performance function means that concentration of DO becomes as close as possible to the target concentration of DO.

\[J = \frac{1}{2} \int_{t_0}^{t_f} \int_{\Omega} (c - c_{obj})^T Q (c - c_{obj}) d\Omega dt, \quad (39)\]

where \(Q, c\) and \(c_{obj}\) are weighting diagonal matrix, the computed DO and the target water DO, respectively. The superscript \(T\) denotes transpose.

#### 4.2 First Order Adjoint Equation

The Lagrange multiplier method is suitable for the minimization problem with constraint condition. The extended performance function \(J^*\) is expressed as follows:

\[J^* = J + \int_{t_0}^{t_f} \int_{\Omega} u^*_i (\dot{u}_i + u_j u_{i,j} + g(\xi + \eta + h),_i - \nu(u_{i,j} + u_{j,i}),_j + fu_i) d\Omega dt
\]

\[+ \int_{t_0}^{t_f} \int_{\Omega} x^* (\dot{\xi} + \{(\xi + h)u_i\},_i) d\Omega dt
\]

\[+ \int_{t_0}^{t_f} \int_{\Omega} c^* (\dot{c} + (u_i c),_i - \kappa c_{ii}) d\Omega dt, \quad (40)\]

Where \(u^*_i, \xi^*\) and \(c^*\) denote the Lagrange multiplier for velocity, water elevation and DO, respectively. The stationary condition which the first variation of \(J^*\) should be zero is needed to minimize the performance.
The gradient of extended performance function with respect to the control variable can be obtained as follows:

\[
\nabla(J^*) = \int_{\Gamma_C} \left( u_i^* u_j n_j + \nu u_i^* j (n_j + n_i) + \xi^* (\xi + h) n_i + c^* c n_i \right) d\Gamma_C dt \quad \text{on} \quad \Gamma_C.
\]

\section{Minimization Technique}

\subsection{Weighted Gradient Method}

In the weighted gradient method, a modified performance function to which a penalty term is added $K$ is used and expressed as follows:

\[
K = J^{*(l)} + \frac{1}{2} \int_{t_0}^{t_f} \left( U^{(l+1)} - U^{(l)} \right)^T W^{(l)} \left( U^{(l+1)} + U^{(l)} \right) dt,
\]

where $l$, $W^{(l)}$ and $U^{(l)}$ are number of iteration, the weighting diagonal matrix and the control velocity, respectively.

In case that the modified performance function is converged to the minimum value, the penalty term will be zero. To minimize the modified performance function is equal to minimize the extended performance function. Let $U$ be the control velocity, then the following equation holds:

\[
\delta K = 0,
\]

The control velocity is updated by following equation:

\[
\nabla(J^*) = -W^{(l)} \left( U^{(l+1)} - U^{(l)} \right).
\]

\subsection{Algorithm of Weighted Gradient Method}

The algorithm of the weighted gradient method is shown as follows:

\begin{enumerate}
\item Chose the initial control velocity $U^{(l)}$, and set the number of iteration $l$ to 0.
\end{enumerate}
Step 2. Solve the state variables $u_i^{(l)}$, $\xi^{(l)}$ and $c^{(l)}$ using Eq.(1), Eq.(2) and Eq.(10).

Step 3. Compute the initial performance function $J^{(l)}$.

Step 4. Solve $u_i^{*^{(l)}}$, $\xi^{*^{(l)}}$ and $c^{*^{(l)}}$ using Eq.(42), Eq.(43) and Eq.(44).

Step 5. Solve the gradient $g^{(l)}$ using Eq.(52)

Step 6. Update the control velocity $U^{(l)}$ using Eq.(55)

Step 7. Solve the state variables $u_i^{(l+1)}$, $\xi^{(l+1)}$ and $c^{(l+1)}$ with the control velocity using Eq.(1), Eq.(2) and Eq.(10).

Step 8. Compute the performance function $J^{(l+1)}$.

Step 9. Solve $u_i^{*^{(l+1)}}$, $\xi^{*^{(l+1)}}$ and $c^{*^{(l+1)}}$ using Eq.(42), Eq.(43) and Eq.(44).

Step 10. Solve the gradient $g^{(l+1)}$ using Eq.(52)

Step 11. Update the control velocity $U^{(l+1)}$ using Eq.(55)

Step 12. Check the convergence; if $\|U^{(l+1)} - U^{(l)}\| < 10^{-6}$ then stop, else go to step 13.

Step 13. Update a weighting parameter $W^{l}$;
if $J^{(l+1)} < J^{(l)}$, then set $W^{(l+1)} = 0.9 W^{(l)}$ and go to step 4
else $W^{(l+1)} = 2.0 W^{(l)}$ and go to step 7.

6 Numerical Study

In this paper, the optimal control of dissolved oxygen (DO) is carried out. The computational model is shown in Fig.4. As an initial condition, the concentration of DO is assumed to be constant which is $2.0 (mg/l)$ at the whole domain. The concentration of DO is given $6.0 (mg/l)$ on the boundary $(a)$. The water depth is $2[m]$ at the whole domain. Time increment $\Delta t$ is $0.5[sec]$ and total time step is 1000. Boundaries $(a)$ and $(b)$ are the inflow boundary and the downstream boundary, respectively. In inverse anarysis line $(a)$ becomes control boundary. Fig.5 shows the target concentratin of DO at the objective point.

This optimal control problem is to find an optimal control velocity on the control boundary $(a)$ so as to minimize performance function. The weighted gradient method is applied as minimization technique. The finite element mesh has 357 nodes and 600 elements.

Fig.3 Finite Element Mesh
7 Numerical Results

Fig. 6 shows the performance function. The weighted gradient method is converged at 25 iteration. Iteration is rearranged in calculation time, weighted gradient method is converged at 56 minutes. Fig. 7 is control velocity on control boundary. Fig. 8 is the concentration of DO at objective point. Therefore, the optimal control of DO in shallow water flow is successful.

8 Conclusion

In this paper, the optimal control of concentration of DO in the shallow water flow is presented. The control velocity can be derived so as to minimize the performance function using the weighted gradient method. As a numerical study, the optimal control of simple computational model is shown. The target concentration of DO can be obtained by control velocity. As for the future works, it is necessary to apply the Newton minimization algorithm using the BFGS method.

Reference


