Parameter Identification of Water Elevation of Tide Using Sensitivity Analysis Method

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Abstract

The purpose of this study is to identify four amplitudes of water elevation of tide using the sensitivity analysis method. The tide at the sea near Japanese coast consists of four constituents. Accidents at the sea or wave disasters, etc. can be prevented beforehand and damage can be reduced by analyzing tide. A parameter identification method is used to find optimal parameter so as to minimize the performance function. The performance function consists of the square sum of the discrepancy between the computed and the observed water elevation. The performance function is differentiated with respect to the amplitude to obtain the sensitivity. As a minimization technique, the weighted gradient method is applied. The amplitude of water elevation is obtained. As the state equation, the nonlinear shallow water equation is used. The Crank-Nicolson method is employed as the temporal discretization and the finite element method is applied as the spatial discretization. The same order interpolation based on the bubble function interpolation is adopted for velocity and water elevation, and the stabilized parameter is introduced. As numerical study in case 1, the parameter identification of the amplitudes of water elevation is carried out. In case 2, the parameters identification in Tokyo bay is carried out.

Keywords: Shallow Water Equation, Finite Element Method, Bubble Function Interpolation, Crank-Nicolson Method, Sensitivity Analysis Method

1 Introduction

Various natural phenomena are repeatedly occurred in the ocean area, which covers over 70 percent of the earth. For example, there is a tide caused by gravitation by the sun and the moon. The tide strongly influences the things that lives in the sea. Also, the tide is important for the fishery industry. Tokyo bay have some ports, Tokyo port, Chiba port, Kawasaki port and so on. The analysis of tide in Tokyo bay is related to our life because there are a lot of fishery industry, maritime industry, and so on.

In the present study, the phenomenon is forecast by the numerical analysis. The tide is analyzed by identifying the amplitude of the water elevation. As the state equation, the nonlinear shallow water equation is used. The Crank-Nicolson method is employed to the temporal discretization and the finite element method is applied to the spatial discretization. The same order interpolation that uses the bubble function interpolation is used for both velocity and water elevation. For the parameter identification technique, sensitivity analysis method is employed. It is necessary to take a lot of time steps and computational time. The sensitivity analysis method is easy to
get the gradient. The initial condition is given to the sensitivity equation, and the sensitivity equation can be solved. To solve the problem using the sensitivity analysis method, the performance function should be defined. The performance function consists of the square sum of the discrepancy between the computed and the target water elevation. The computed water elevation is going to be close to the target water elevation by iterative computation. It is important to find the optimal parameters so as to minimize the performance function. Water elevation is updated by the weighted gradient method in the iterative calculation. The weighted gradient method is applied as the minimization technique. In this study, four constituents (M2, S2, O1, K1) are used for the boundary condition, therefore, four parameters are identified. The purpose of this study is the parameter identification of tidal constituents.

2 Tidal Constituents

Tidal fluctuation can be expressed in the sum of trigonometric series. Tidal constituents can be classified into 390 constituents by the period. Main four tidal constituents which have important role are used for computing the prediction of tidal fluctuation.

Main four tidal constituents are;
1. Principal lunar semidurnal tides (M2) ; the tide by diurnal rotation of the moon.
2. Principal solar semidurnal tides (S2) ; the tide by diurnal rotation of the sun.
3. Lunisolar tides (K1) ; the tide by relative position of the sun and the moon.
4. Principal lunar diurnal tides (O1) ; the tide by diurnal rotation of the moon.

The magnitudes of tide-generating force are M2, S2, O1 and K1 in the order. The periods of M2, K1, S2 and O1 are 12h25m, 23h56m, 12h00m and 25h49m respectively. Tidal fluctuation is given using the main tidal constituents as follows:

\[
\eta = \sum_{k=1}^{4} A_k \sin \left( \frac{2\pi t}{T_i} + \kappa \right)
\]  

3 State Equation

3.1 Shallow Water Equation

The nonlinear shallow water equation are employed as the state equations. The shallow water equations consist of momentum equation and continuity equation, as follows;

\[
\dot{u}_i + u_j u_{i,j} + g(\xi + \eta + H)_{,i} - \nu(u_{i,j} + u_{j,i})_{,j} + fu_i = 0 \quad \text{in} \quad \Omega, \tag{2}
\]

\[
\dot{\eta} + \{(\eta + H)u_i\}_{,i} = 0 \quad \text{in} \quad \Omega, \tag{3}
\]

where \( u_i, g, \xi, \eta, H, \nu, \) and \( f \) denote velocity, gravity acceleration, bed elevation, water elevation, water depth, the kinetic eddy viscosity coefficient and bottom friction, respectively. \( \Omega \) means an computational domain. The
coordinate system is shown in Figure 1.

![XYZ Coordinate system](image)

Figure 1: XYZ Coordinate system

boundary conditions are given as follows:

\[
\begin{align*}
    u_i &= \hat{u}_i \quad \text{on} \quad \Gamma_d, \\
    \eta &= A_k \sin \left( \frac{2\pi t}{T_i} + \kappa \right) = \hat{\eta} \quad \text{on} \quad \Gamma_d, \\
    u_i &= u_i n_i = \hat{u}_n \quad \text{on} \quad \Gamma_n,
\end{align*}
\]

where the boundaries \( \Gamma_d \) and \( \Gamma_n \) are the Dirichlet and Neumann boundaries, respectively. Amplitude of sinusoidal wave is denoted by \( A_k \) and applied as the water elevation in eq.(4), and \( n_i \) is unit vector of outward normal to \( \Gamma \).

Initial conditions are expressed as follows:

\[
\begin{align*}
    u_i(t_0) &= \hat{u}_i(t_0) \quad \text{in} \quad \Omega, \\
    \eta(t_0) &= \hat{\eta}(t_0) \quad \text{in} \quad \Omega.
\end{align*}
\]

4 Discretization Technique

4.1 Spatial Discretization

For the discretization, the technique described in Matsumoto et al. [2003] is used. The state equations are discretized by using the finite element method based on the bubble function interpolation in space. The bubble function interpolation is applied to the velocity and water elevation fields.
\[ u_i = \Phi_1 u_{i1} + \Phi_2 u_{i2} + \Phi_3 u_{i3} + \Phi_4 \bar{u}_{i4}, \quad (9) \]

\[ \bar{u}_{i4} = u_{i4} - \frac{1}{3} (u_{i1} + u_{i2} + u_{i3}), \quad (10) \]

\[ \eta = \Phi_1 \eta_1 + \Phi_2 \eta_2 + \Phi_3 \eta_3 + \Phi_4 \bar{\eta}_4, \quad (11) \]

\[ \bar{\eta}_4 = \eta_4 - \frac{1}{3} (\eta_1 + \eta_2 + \eta_3), \quad (12) \]

\[ \Phi_1 = L_1, \quad \Phi_2 = L_2, \quad \Phi_3 = L_3, \quad \Phi_4 = 27L_1L_2L_3, \quad (13) \]

![Figure 2: Bubble Function Element](image)

The spatial discretized forms of the momentum and continuity equations are expressed as:

\[ M_{\alpha\beta} \dot{u}_{\beta i} + A_{\alpha\beta\gamma j} u_{\gamma j} u_{\beta i} + g S_{\alpha\beta i} (\eta_\beta + \xi_\beta + H_\beta) + \nu D_{\alpha i \beta j} u_{\beta j} + f M_{\alpha\beta} u_{\beta i} = 0, \quad (14) \]

\[ M_{\alpha\beta} \dot{\eta}_\beta + A_{\alpha\beta\gamma i} (\eta_\beta + H_\beta) u_{\gamma i} + A_{\alpha\beta\gamma i} u_{\beta i} \eta_\gamma = 0, \quad (15) \]

where

\[ M_{\alpha\beta} = \int_\Omega (\Phi_\alpha \Phi_\beta) d\Omega, \quad (16) \]

\[ A_{\alpha\beta\gamma j} = \int_\Omega (\Phi_\alpha \Phi_\beta \Phi_\gamma,j) d\Omega, \quad (17) \]

\[ S_{\alpha\beta i} = \int_\Omega (\Phi_\alpha \Phi_\beta,i) d\Omega, \quad (18) \]

\[ D_{\alpha i \beta j} = \int_\Omega \{(\Phi_\alpha,k \Phi_\beta,l) \delta_{ij} + (\Phi_\alpha,j + \Phi_\beta,i)\} d\Omega. \quad (19) \]

### 4.2 Temporal Discretization

The Crank-Nicolson method is applied to the finite element equations in time. The finite element equations are transformed into the following form:
\[
M_{\alpha\beta} \frac{u_{n+1}^{\beta} - u_n^{\beta}}{\Delta t} + A_{\alpha\beta\gamma} u_{n+1}^{\beta} u_{n+1}^{\gamma} + g S_{\alpha\beta} \eta_{n+\frac{1}{2}}^{\beta} + \xi_{\beta} + H_{\beta} + \nu D_{\alpha\beta\gamma} u_{n+\frac{1}{2}}^{\beta} + f M_{\alpha\beta} u_{n+\frac{1}{2}}^{\beta} = 0, \quad (20)
\]

\[
M_{\alpha\beta} \frac{\eta_{n+1}^{\beta} - \eta_{n}^{\beta}}{\Delta t} + A_{\alpha\beta\gamma} (\eta_{n+\frac{1}{2}}^{\beta} + H_{\beta}) u_{n+1}^{\gamma} + A_{\alpha\beta\gamma} u_{n+\frac{1}{2}}^{\beta} \eta_{n} = 0, \quad (21)
\]

where

\[
u_{n+\frac{1}{2}}^{\beta} = \frac{1}{2} (u_{n+1}^{\beta} + u_n^{\beta}), \quad \eta_{n+\frac{1}{2}}^{\beta} = \frac{1}{2} (\eta_{n+1}^{\beta} + \eta_n^{\beta}). \quad (22)
\]

5 Formulation

5.1 Performance Function

The purpose of this study is to identify the amplitude of the water elevation of the tide. The performance function is introduced, which consists of the square sum of the discrepancy between the computed and the observed water elevation. A parameter identification is to find optimal parameters so as to minimize the performance function \(J\), which is expressed as follows:

\[
J = \frac{1}{2} \int_{t_0}^{t_f} (\eta_{n} - \eta_{n}^{\text{obj}}) Q_{\alpha\beta} (\eta_{\beta} - \eta_{\beta}^{\text{obj}}) dt, \quad (23)
\]

where \(\eta_n\) and \(\eta_n^{\text{obj}}\) are the computed and observed water elevation, respectively at the node \(\alpha\) and \(t_0\) and \(t_f\) mean the initial and final times, respectively.

5.2 Sensitivity analysis Method

The sensitivity analysis method is applied to identify the parameters. It is necessary to derive the gradient of the performance function with respect to the amplitude so as to minimize the performance function. The gradient can be obtained to differentiate the performance function with respect to \(A_k\):

\[
\frac{\partial J}{\partial A_k} = \int_{t_0}^{t_f} (\eta_{n} - \eta_{n}^{\text{obj}}) Q_{\alpha\beta} \frac{\partial \eta_{\beta}^{\text{obj}}}{\partial A_k} dt. \quad (24)
\]

where \(A_k\) is the amplitude of constituents of tide. The state equation is differentiate by the amplitude \(A_k\) to derive the sensitivity equation as follows;

\[
\langle \text{Momentum Equation} \rangle
\]

\[
\frac{1}{\Delta t} M_{\alpha\beta} \frac{\partial}{\partial A_k} (u_{n+1}^{\beta} - u_n^{\beta}) + A_{\alpha\beta\gamma} u_{n+1}^{\gamma} \frac{\partial u_{n+1}^{\gamma}}{\partial A_k} + g S_{\alpha\beta} \frac{\partial \eta_{n+\frac{1}{2}}^{\beta}}{\partial A_k} + \xi_{\beta} + H_{\beta} + \nu D_{\alpha\beta\gamma} \frac{\partial u_{n+\frac{1}{2}}^{\beta}}{\partial A_k} + f M_{\alpha\beta} \frac{\partial u_{n+\frac{1}{2}}^{\beta}}{\partial A_k} = 0, \quad (25)
\]
\[
\frac{1}{\Delta t} M_{\alpha\beta} \frac{\partial}{\partial A_k} (\eta_{\beta i}^{n+1} - \eta_{\beta i}^n) + A_{\alpha\beta\gamma} \frac{\partial u_{\alpha \beta}}{\partial A_k} \eta_{\gamma i}^{n+1} + A_{\alpha\beta\gamma} u_{\beta} \frac{\partial \eta_{\gamma i}^{n+1}}{\partial A_k} + A_{\alpha\beta\gamma} \frac{\partial (\eta_\beta + H_\beta)}{\partial A_k} u_{\gamma i}^{n+1} + A_{\alpha\beta\gamma} (\eta_\beta + H_\beta) \frac{\partial u_{\gamma i}^{n+1}}{\partial A_k} = 0,
\]

(26)

The boundary conditions for the sensitivity analysis can be derived by differentiating the boundary conditions of the state equations:

\[
\frac{\partial u_i}{\partial A_k} = \frac{\partial \hat{u}_i}{\partial A_k} \quad \text{on } \Gamma_d,
\]

(27)

\[
\frac{\partial \eta}{\partial A_k} = \sin(\frac{2\pi t}{T_i} + \kappa) = \frac{\partial \hat{\eta}}{\partial A_k} \quad \text{on } \Gamma_d,
\]

(28)

\[
\frac{\partial u_i}{\partial A_k} = \frac{\partial u_i}{\partial A_k} n_i = \frac{\partial \hat{u}_n}{\partial A_k} \quad \text{on } \Gamma_n.
\]

(29)

The initial conditions for the sensitivity analysis can be also obtained as follows;

\[
\frac{\partial u_i(t_0)}{\partial A_k} = 0 \quad \text{in } \Omega,
\]

(30)

\[
\frac{\partial \eta(t_0)}{\partial A_k} = 0 \quad \text{in } \Omega.
\]

(31)

6 Minimization Technique

6.1 Weighted Gradient Method

As the minimization technique, the weighted gradient method is applied. In the gradient method, a modified performance function \( K \) in which a penalty term is added is introduced as follows;

\[
K^{(l)} = J^{(l)} + \frac{1}{2} \left( A_k^{(l+1)} - A_k^{(l)} \right) W^{(l)} \left( A_k^{(l+1)} - A_k^{(l)} \right),
\]

(32)

where \( l \) and \( W^{(l)} \) are number of iteration and the weighting parameter. In case that the modified performance function is converged to the minimum value, the penalty term will be zero. To minimize the modified performance function is equal to minimize the performance function. Let \( A_k \) be the amplitude of water elevation, then the following equation holds:

\[
\frac{\partial K^{(l)}}{\partial A_k} = 0,
\]

The amplitude of water elevation is updated by the following equation.
6.2 Algorithm of Sensitivity Analysis Method

The algorithm of the sensitivity analysis method is shown as follows:

Step 1. Chose a parameter $A_k^{(0)}$.
Step 2. Solve the state variables $u_i^{(l)}$ and $\eta_i^{(l)}$ using eqs.(2) and (3) using $A_k^{(l)}$.
Step 3. Compute the performance function $J^{(l)}$.
Step 4. Solve the sensitivity equation to get $\frac{\partial u_i^{(l)}}{\partial A_k}$ and $\frac{\partial \eta_i^{(l)}}{\partial A_k}$ using eqs.(25) and (26).
Step 5. Solve the gradient $\frac{\partial J}{\partial A_k}$ using eq.(24).
Step 6. Update the amplitude of water elevation $A_k^{(l)}$ using eq.(34).
Step 7. Solve the state variables $u_i^{(l+1)}$ and $\eta_i^{(l+1)}$ using eqs.(2) and (3).
Step 8. Compute the performance function $J^{(l+1)}$.
Step 9. If $|J^{(l+1)} - J^{(l)}| < \epsilon$ then stop, else go to step.10.
Step 10. Update a weighting parameter $W^{(l)}$; if $|J^{(l+1)} - J^{(l)}| < 0$, then set $W^{(l+1)} = 0.9W^{(l)}$ and go to Step 4. else $W^{(l+1)} = W^{(l)}$ go to Step 6.

7 Verification

In this study, the parameter identification of constituents of sinusoidal wave is carried out. The amplitudes of four waves are identified. The computational domain is shown in Fig.3. The water depth is 10[m] in the whole domain. Time increment $\Delta t$ is 0.002[s] and total time step is 3000. The objective points are set on the central points of the computational domain. On $\Gamma_D$, the water elevation $\eta$ is given as shown in Fig.(3). The sinusoidal wave is described in eq.(4). The amplitudes of target value are set to 0.1 [m], 0.2 [m], 0.3 [m] and 0.4 [m] respectively. The initial amplitudes are set to 0.11 [m], 0.21 [m], 0.31 [m] and 0.41 [m] respectively. The finite element mesh which has 1203 nodes and 1600 elements is shown in Fig.4.

$$A_k^{(l+1)} = A_k^{(l)} - W^{(l)} \cdot \frac{\partial J^{(l)}}{\partial A_k}$$ (34)
Figure 4: Finite Element Mesh

Figure 5: Variation of performance function

Figure 6: Variation of amplitude1 of water elevation

Figure 7: Variation of amplitude2 of water elevation
As numerical result in case 1, four amplitudes of water elevation can be identified. The variation of the performance function is shown in Fig.5. The variation of four amplitudes of water elevation are shown in Fig.6~9 respectively.

8 Numerical Studies

In this study, the parameter identification of tidal constituents on Tokyo bay is carried out. Time increment $\Delta t$ is 3600[s] and total time step is 10000. The objective points are set on Tokyo port. The finite element mesh which has 2393 nodes and 4435 elements is shown in Fig.10. The water depth in Tokyo bay is shown in Fig.11.
8.1 Case 1

The amplitudes of four waves are identified. As numerical result in case 1, four amplitudes of water elevation can be identified. The variation of the performance function is shown in Fig.12. The variation of four amplitudes of water elevation are shown in Fig.13～16 respectively. Identified data is shown in Fig.17.
Figure 13: Variation of amplitude of M2

Figure 14: Variation of amplitude of S2

Figure 15: Variation of amplitude of O1

Figure 16: Variation of amplitude of K1
8.2 Case 2

The phase lag of four waves are identified. As numerical result in case 2, four phase lag of water elevation can be identified. The variation of the performance function is shown in Fig.18. The variation of four phase lag of water elevation are shown in Fig.19~22 respectively. Identified data is shown in Fig.23.
Figure 19: Variation of phase lag of M2

Figure 20: Variation of phase lag of S2

Figure 21: Variation of phase lag of O1

Figure 22: Variation of phase lag of K1
8.3 Case 3

The amplitudes and phase lag of four waves are identified. As numerical result in case 3, four amplitudes of water elevation can be identified. The variation of the performance function is shown in Fig.24. The variation of four amplitudes of water elevation are shown in Fig.25~28 respectively. The variation of four phase lag of water elevation are shown in Fig.29~32 respectively. Identified data is shown in Fig.33.
Figure 25: Variation of amplitude of M2

Figure 26: Variation of amplitude of S2

Figure 27: Variation of amplitude of O1

Figure 28: Variation of amplitude of K1
Figure 29: Variation of phase lag of M2

Figure 30: Variation of phase lag of S2

Figure 31: Variation of phase lag of O1

Figure 32: Variation of phase lag of K1
Figure 33: Actual observed data and identified data

9 Conclusion

The parameter identification using the sensitivity analysis method is presented for the determination of four constituents of water elevation. Four constituents of water elevation can be identified using the finite element method and the sensitivity analysis method. Four constituents of water elevation are found so as to minimize the performance function. Four constituents of water elevation are converged to the target values from the initial values. As a result, the parameter identification using the sensitivity analysis method is shown to be effective for the determination of four constituents of water elevation. Also we can obtained the great result of accuracy by identifying the amplitudes and phase lag.

References

