Application of Reduced Kalman Filter Finite Element Method to the Ground

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Abstract
The purpose of this study is to investigate estimation of acceleration of the ground using the Kalman filter finite element method. From data of acceleration observed in the site, arbitrary data of acceleration is estimated. In this paper, as the state equation, the equilibrium of stress equation, the strain-displacement equation, and the stress-strain equation are used. For spatial discretization, the finite element method and the Galerkin method is used as an interpolation method. For temporal discretization, the Newmark β method is applied. The Kalman filter finite element method is the combination of the Kalman filter and the finite element method. This method can estimate from noisy observation in the site. However, long computational time is required for computation by the method. Then, to reduce the computational time, the computational domain is divided into two parts, the main domain and the subsidiary domain. In the main domain, filtering procedures are carried out, whereas only deterministic process is taken for the variables in the subsidiary domain. Eliminating the state variables in the subsidiary domain, the drastically efficient computation is carried out. In this paper, this method is called the Reduced Kalman Filter Finite Element Method. As the numerical study, this method is applied to Ohyorogi tunnel site. The site is located in Mt.Ohyorogi in Hiroshima prefecture, Japan. The blasting examination was carried out on August 26th, 2009. Then accelerations are measured at two points by the accelerometer. These obtained data are used as observation and reference data. The acceleration is estimated by the presented method using observation data. The estimation value is compared with reference data at estimation point.

key words: Finite element method, Reduced Kalman filter finite element method, Balance of stress equation, Strain-displacement equation, Stress-strain equation, Ohyorogi tunnel

1 INTRODUCTION
In recent year, the finite element method has developed rapidly in the world by the remarkable development of computer. Then, it is possible to analyze a mountain, a construction site and so on by the computer if there are observed values. The finite element method is a technique to solve entire behavior as approximate values after the object with complex shape and character divided small until simplify and numerical results in each parts are added. This technique aimed to reproduce the observed phenomenon by using an adequate model, initial and boundary condition. However, if an entire mountain and a construction site are observed, it needs costs and times, and it is difficult to observe according to the environment. Therefore, it is necessary to estimate entire state of value with a little observed data. Then the Kalman filter is used as a new approach.
The Kalman filter is one of a filtering theory based on a probability process. It has been the filtering algorithm presented by Kalman and Bucy 1960’s which is based on the theory is state space model and composed of two systems, which are expressed by the system and observation equation. In these equations, the system and observation noises are included. It is defined as the system noise that the difference between actually natural phenomenon and state value by the basic equation. It is used widely in various fields. For example, finance engineering, space engineering, communication engineering, civil engineering and so on. But it can estimate the state only in time direction. Then, by combining the Kalman filter and the finite element method, the Kalman filter finite element method can estimate state value not only in time direction. This is a technique to estimate accelerate using observed values which are corrected based on the stochastic process. So it becomes possible to estimate entire state of value from a little observed data. However, This method takes long computational time, because it use large matrix. To reduce computational time, domain decomposition is performed. The entire computational domain can be divided into two domains i.e. one is the main domain which is carried out the main computational and the other is the subsidiary domain which is performed auxiliary computational by the finite element method. The variables the subsidiary domain are used only for the computation by the finite element method. The state variables in the subsidiary domain are used for the deterministic process. In this paper, as the numerical example, this method is applied to the estimation of acceleration in the Ohyorogi tunnel site in Hiroshima prefecture, Japan.

2 STATE EQUATION

In this paper, usual indicial notation and summation convention with repeated indices are used. The equilibrium of stress equation is expressed as,

\[ \sigma_{ij,j} - \rho b_i - \rho \ddot{u}_i = 0, \]  

(1)

where \( \sigma_{ij}, \ b_i, \ \rho, \ \ddot{u}_i \) denote total stress, body force, density of the ground and acceleration. The strain - displacement equation can be described in the following form

\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \]  

(2)

where \( \varepsilon_{ij} \) and \( u_i \) are strain and displacement respectively. The stress - strain equation is

\[ \sigma_{ij} = D_{ijkl}\varepsilon_{kl}, \]  

(3)

where \( D_{ijkl} \) express coefficient of elastic stress - strain relation and can be written as,

\[ D_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \]  

(4)

\[ \lambda = \frac{\nu}{(1-2\nu)(1+\nu)}, \]  

(5)

\[ \mu = \frac{E}{2(1+\nu)}, \]  

(6)

where \( \lambda \) and \( \mu \) are Lame’s constant. \( \delta_{ij} \) is Kronecker’s delta. \( E \) and \( \nu \) are the elastic modulus and Poisson ratio respectively.

3 BOUNDARY CONDITION

In this paper, the boundary \( S \) is divided into \( S_1 \) and \( S_2 \). On \( S_1 \) boundary, displacement is specified and on \( S_2 \) boundary, surface force \( t_i \) is given.
\[ u = \tilde{u}_i \quad \text{on} \quad S_1 \]  
\[ t_i = \sigma_i n_j = \hat{t}_i \quad \text{on} \quad S_2 \]

where \( \tilde{u}_i \) and \( \hat{t}_i \) are specified values on the boundary, \( n_j \) is the external unit normal vector to the boundary.

### 4 Finite Element Equation

In this paper, for domain discretization, the finite element method is employed and the Galerkin method is used as an interpolation method. The discretized equation with linear tetrahedral element is obtained as follow

\[ M_{\alpha i\beta k} \ddot{u}_{\beta k} + K_{\alpha i\beta k} u_{\beta k} = \hat{\Gamma}_{\alpha i}, \quad (9) \]

in addition, considering the effect of damping, eq. (9) can be expressed as

\[ M_{\alpha i\beta k} \ddot{u}_{\beta k} + C_{\alpha i\beta k} \dot{u}_{\beta k} + K_{\alpha i\beta k} u_{\beta k} = \hat{\Gamma}_{\alpha i}, \quad (10) \]

\[ M_{\alpha i\beta k} = \rho \int_V \delta_{ik} (N_\alpha N_\beta) \, dV, \quad (11) \]

\[ C_{\alpha i\beta k} = \alpha_0 M_{\alpha i\beta k} + \alpha_1 K_{\alpha i\beta k}, \quad (12) \]

\[ K_{\alpha i\beta k} = \int_V (N_{\alpha,j} D_{ijkl} N_{\beta,l}) \, dV, \quad (13) \]

\[ \hat{\Gamma}_{\alpha i} = \rho \int_V (N_\alpha \hat{b}_i) \, dV - \int_S (N_\alpha \hat{t}_i) \, dS, \quad (14) \]

in which \( N_\alpha \) is the shape function. For damping, eq.(12) is assumed, where \( \alpha_0, \alpha_1 \) are empirically used damping coefficients.

### 5 The Newmark \( \beta \) Method

In this paper, the Newmark \( \beta \) method is applied to the finite element equation, in which, the equation at the time \((n+1)\) is expressed as follow,

\[ M_{\alpha i\beta k} \ddot{u}_{\beta k}^{(n+1)} + C_{\alpha i\beta k} \dot{u}_{\beta k}^{(n+1)} + K_{\alpha i\beta k} u_{\beta k}^{(n+1)} = \hat{\Gamma}_{\alpha i}, \quad (15) \]

in the method, velocity and displacement at the time \((n+1)\) can be expressed as follows

\[ \ddot{u}_{\beta k}^{(n+1)} = \dot{u}_{\beta k}^{(n)} + \Delta t \ddot{u}_{\beta k}^{(n+1)} + \gamma \Delta t (\ddot{u}_{\beta k}^{(n+1)} - \dot{u}_{\beta k}^{(n)}), \quad (16) \]

\[ \ddot{u}_{\beta k}^{(n+1)} = \ddot{u}_{\beta k}^{(n)} + \Delta t \ddot{u}_{\beta k}^{(n)} + \frac{(\Delta t)^2}{2} \ddot{u}_{\beta k}^{(n)} + \beta (\Delta t)^2 (\ddot{u}_{\beta k}^{(n+1)} - \ddot{u}_{\beta k}^{(n)}), \quad (17) \]

eqs. (16) and (17) are submitted into eq.(15), the following equation can be derived as follows

\[ \ddot{u}_{\beta k}^{(n+1)} = D_{\alpha i\beta k}^{-1} (\ddot{\Gamma}_{\alpha i} - E_{\alpha i\beta k} \ddot{u}_{\beta k}^{(n)} - J_{\alpha i\beta k} \dot{u}_{\beta k}^{(n)} - K_{\alpha i\beta k} u_{\beta k}^{(n)}), \quad (18) \]

where \( D_{\alpha i\beta k}, E_{\alpha i\beta k} \) and \( F_{\alpha i\beta k} \) are written as,
\[ D_{\alpha i\beta k} = M_{\alpha i\beta k} + \Delta t \gamma C_{\alpha i\beta k} + (\Delta t)^2 \left( \frac{1}{2} - \beta \right) K_{\alpha i\beta k}, \]  
(19) 
\[ E_{\alpha i\beta k} = \Delta t (1 - \gamma) C_{\alpha i\beta k} + (\Delta t)^2 \left( \frac{1}{2} - \beta \right) K_{\alpha i\beta k}, \]  
(20) 
\[ J_{\alpha i\beta k} = C_{\alpha i\beta k} + \Delta t K_{\alpha i\beta k}, \]  
(21)

in which \( M_{\alpha i\beta k} \), \( C_{\alpha i\beta k} \) and \( K_{\alpha i\beta k} \) are expressed in eqs.(11), (12) and (13), respectively.

6 THE KALMAN FILTER

Natural phenomena can be usually expressed by the following equation;

\[ x_{k+1} = F_k x_k + G_k w_k, \]  
(22)

where eq. (22) is called the system equation, in which \( x_k \) denote state variable at time \( k \), \( F_k \) and \( G_k \) are state transition and driving matrices, \( w_k \) is system noise included at the occasion the system is discretized. The observation \( y_k \) cannot be obtain at a whole domain, however, at some limited observation points, thus

\[ y_k = H_k x_k + v_k, \]  
(23)

where \( H \) means the coefficient matrix which expressed the correspondence between the observation and state variable and \( v_k \) is the observation noise, respectively. Both \( w_k \) and \( v_k \) are assumed;

\[ w_k \sim N(0, Q), \]  
(24) 
\[ v_k \sim N(0, R), \]  
(25)

where \( N(a, A) \) represents the normal distribution with mean \( a \) and \( A \). It is also assumed that;

\[ E\{ w_k, v_k \} = 0, \]  
(26)

where \( E\{ \} \) is an expectation operator. Observation data are obtain data each time as;

\[ Y_k = [y_1, y_2, \cdots y_k], \]  
(27)

The optimal estimate \( \hat{x}_k \) is an expectation of \( x_k \) and giving observation data \( Y_k \),

\[ \hat{x}_k = E\{ x_k \} | Y_k \}, \]  
(28)

The initial condition is given;

\[ \hat{x}_0 = \hat{x}_0, \]  
(29)

where \( \hat{x}_0 \) is a specified value. The state estimate \( \hat{x}_k \) using the new measurement \( y_k \) is obtained as follows;

\[ \hat{x}_k = x^*_k + K_k [y_k - H x^*_k], \]  
(30)

where \( K_k \) is referred to as the Kalman gain, which will be given later in eq.(37), and \( x^*_k \) is a priority estimation,
\[ x_k^* = E\{x_k|Y_{k-1}\}, \] (31)

The estimated error covariance \( P_k \) is derived as

\[
P_k = \text{cov}\{x_k|Y_k\} \\
= E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\} \\
= (I - K_kH)\Gamma_k(I - HK_k^T) + K_kR_kK_k^T \\
= \Gamma_k - K_k\Gamma_kH_k - \Gamma_kH_k^T K_k + K_k(\Gamma_kH_k^T + R_k)K_k^T \\
= (I - K_kH)\Gamma_k, \quad (32)
\]

where covariance \( \Gamma_k \) is defined as

\[
\Gamma_k = \text{cov}\{x_k|Y_{k-1}\} \\
= E\{(x_k - x_k^*)(x_k - x_k^*)^T\} \\
= E\{(F_k\hat{x}_{k-1} + G_{k-1}w_{k-1} - F_k\hat{x}_{k-1})(F_k\hat{x}_{k-1} + G_{k-1}w_{k-1} - F_k\hat{x}_{k-1})^T\} \\
= F_k\Gamma_{k-1}F_k^T + G_{k-1}Q_{k-1}G_{k-1}^T, \quad (33)
\]

where \( \Gamma_k \) is called predicted error covariance matrix. Initial condition of the matrix is given;

\[
\Gamma_0 = \hat{\Gamma}_0, \quad (34)
\]

where \( \Gamma_0 \) is a specified value. One step ahead prediction of the state variable is obtained;

\[
\hat{x}_{k+1} = F_k\hat{x}_k + B, \quad (35)
\]

Then, it is found that

\[
P_{k+1} = F_k\Gamma_k F_k^T + Q, \quad (36)
\]

the Kalman gain can be determined by

\[
K_k = \Gamma_kH_k^T (R + H\Gamma_kH_k^T)^{-1}, \quad (37)
\]

and the predicted error covariance \( \Gamma_{k+1} \) can be derived as follows;

\[
\Gamma_{k+1} = F_k\Gamma_k F_k^T + G_kQG_k^T \quad (38)
\]

To apply the finite element method to the Kalman filter, the finite element equation is used as the state transition matrix. From the finite element equation eq.(18), the state transition matrix is given as;

\[
\tilde{u}_{\beta k} = F_{\alpha i \beta k}\tilde{u}_{\beta k}^n + \psi_{\alpha i} \quad (39)
\]

\[
F_{\alpha i \beta k} = -D_{\alpha i \beta k}^{-1}\{\Delta t(1 - \gamma)C_{\alpha i \beta k} + (\Delta t)^2(\frac{1}{2} - \beta)K_{\alpha i \beta k}\}, \quad (40)
\]

\[
\psi_{\alpha i} = -D_{\alpha i \beta k}^{-1}\{J_{\alpha i \beta k}\tilde{u}^n + K_{\alpha i \beta k}\tilde{u}_{\beta k}^n - \hat{\Gamma}_{\alpha i}\}, \quad (41)
\]

where \( C_{\alpha i \beta k}, K_{\alpha i \beta k}, D_{\alpha i \beta k} \) and \( J_{\alpha i \beta k} \) are expressed in eqs.(12), (13), (19) and (21).
Because of calculation of inverse matrix, not element matrix but full matrix must be used by the Kalman filter finite element method. So long computational time is needed. To reduce the computational time, the computational domain is divided into two parts, one is referred to as the main domain and the other is termed the subsidiary domain. The main domain consists of observation points, estimation points and other related points. It is based on assumption that system and observation error covariances at a considering point are not correlated with those at points far from the considering point.

Figure 1 is shown main and related point, in which $dm$ is the maximum length between the considering point and the correlated points. The points inside the circle of radius $dm$ are referred to as the related points. $r_i$ is shown distance between main point and related points. The main domain includes the nodal points at which the Kalman filter is applied. Figure 2 is shown the spatial distribution of error covariances that are assumed to be expresses by the quartic spline function as,

$$S = \bar{S}\{1.0 - 6.0 \left( \frac{r_i}{dm} \right)^2 + 8.0 \left( \frac{r_i}{dm} \right)^3 - 3.0 \left( \frac{r_i}{dm} \right)^4 \}$$

(42)

where $S$ stands for $Q$ and $R$, and $\bar{S}$ is the intensity of the covariance. The observation data is obtained by normal measurements always show the spatial distribution shown eq.(43).

Then $x_k$ and $z_k$ are assumed main domain and the other state variable, respectively. The transition matrix can be divided into four parts and the system equation, shown in eq.(22), can be split into the following equation.

$$\begin{bmatrix} x_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} F_k & L_k \\ M_k & N_k \end{bmatrix} \begin{bmatrix} x_k \\ z_k \end{bmatrix} + \begin{bmatrix} G_k w_k \end{bmatrix}$$

(43)

In short, a stochastic process can be applied only to the state variable $x_k$ in the main domain. The state variable $z_k$ in the subsidiary domain is assumed the area without margin of error. Therefore right error term is set 0 in eq.(43). it is used for auxiliary computation by the finite element method. So eq.(22) can be derived as

$$\dot{x}_{k+1} = F_k \dot{x}_k + C_k + G_k w_k,$$

(44)
where $C_k$ is shown as follow

$$C_k = L_k(M_kx_{k-1} + N_{k-1}z_{k-1}),$$

(45)

and $z_k$ can be computed by the last half of eq.(42).

The algorithm of the Kalman filter finite element method is written as;

1. $[\Gamma_0] = [V_0], [\hat{x}^{-1}] = \{\hat{x}^0\}$
2. $[K_k] = [\Gamma_k][H_k]^T([R_k] + [H_k][\Gamma_k][H_k]^T)^{-1}$
3. $[P_k] = ([I] - [K_k][H_k])[\Gamma_k]$
4. $[\Gamma_k] = [F_k][P_k][K_k]^T + [G_k][Q_k][G_k]^T$
5. $\{x_k^\bullet\} = [F_{k-1}]\{x_{k-1}\} + \{\psi_{k-1}\}$
6. $\{\hat{x}_k\} = \{x_k^\bullet\} + [K_k](\{y_k\} - [H_k]\{x_k^\bullet\})$,

where $k$ expresses time step.
8 NUMERICAL STUDIES

In this study, there are two cases. In case 1, the effectiveness the Kalman Filter finite element method is verified. In case 2, The Kalman Filter finite element method is applied Ohyorogi tunnel site.

8.1 Case 1.

In this case, the structure assumed elastic body is computed at two points by the finite element method. Then these points are assumed observation and estimation point, respectively. At observation point, computed data is put noises and it is assumed observation data. At estimation point, computed data is compared with the data that is estimated using observation data by the Kalman Filter finite element method. The finite element mesh is shown in Fig.3. Total number of nodes and elements are 1029 and 4320, respectively. As a boundary condition, right side face is fixed. The others are free condition. The amount of increase at time $\Delta t$ is 0.001(s). Damping coefficient $\alpha_0$ and $\alpha_1$ are set as 0.005 and 0.01. The elastic modulus, Poisson ratio and density of the ground are set as $2.0 \times 10^7 [KN/m^2]$, 0.35 and 2.3[Kg/m$^2$]. As the external force, uniformly distributed load is added to left face. Fig. 4 is image of the addition of the load. Fig.5 show the time history of external force.

![Finite element mesh](image1.png)  
**Figure 3:** Finite element mesh

![External force](image2.png)  
**Figure 4:** External force

![Time history of external force](image3.png)  
**Figure 5:** Time history of external force
8.1.1 Observation data

Generally, observation data is included noises. There are caused at observation and computation and so on. So computed data by the finite element method is put noises. And it is assumed observation data. Observation point on the finite element mesh is shown in Fig. 6. Figs, 7, 8 and 9 show the time history of acceleration in x, y and z-direction in observation point, respectively.

Figure 6: X-Acceleration at observation point

Figure 7: X-Acceleration at observation point

Figure 8: Y-Acceleration at observation point

Figure 9: Z-Acceleration at observation point
8.1.2 Result

Figs 10, 11 and 12 show the comparison of acceleration between computed data by the finite element method and estimation data using the Kalman filter finite element method at estimation point. The noise as shown in Figs 7, 8 and 9 have been clearly filtered by the present method. And the effectiveness the Kalman Filter finite element method is verified.

Figure 10: X-direction comparison

Figure 11: Y-direction comparison

Figure 12: Z-direction comparison
8.2 Case 2.

In this case, actual measured data is used. So Ohyorogi tunnel site is applied. Ohyorogi tunnel site is in Mt.Ohyorogi in Hiroshima prefecture, Japan. These picture and figures shows Ohyorogi tunnel site, the model of it, and estimation (Point1) and observation (Point2) points. The area to be consider about $250 \times 250[m]$, and then the highest point is $205[m]$. Total number of nodes and elements are 4733 and 22649, respectively. As a boundary condition, Vertical direction in each surface is fixed, and Horizontals direction in each surface is free. At the day that is surveyed the site. The tunnel is $150[m]$ long in X-direction. Time increment $\Delta t$ is 0.001[s]. The empirically used damping coefficients $\alpha_0$ and $\alpha_1$ are set as 0.01 and 0.001 respectively. The discretization error depends on the mesh refinement. Thus, almost system errors can be evaluated by the intensity of system error covariance $\bar{S} = 1.0 \times 10^{-3}$ for the computation in this paper. Poisson ratio and density of the ground are set as 0.3 and $2.6 \times 10^3[Kg/m^3]$. This data is referred that the rock quality of this area is the dacite.

Figure 13: Ohyorogi tunnel site

Figure 14: Finite element mesh
Then, finite element mesh is divided into three layers as shown previous Figure. This is referred actual data of geological survey. Where each elastic modulus is from on to $5.0 \times 10^6 [KN/m^2]$, $1.47 \times 10^7 [KN/m^2]$ and $2.47 \times 10^7 [KN/m^2]$ in order. It is referred elastic wave velocity in the data. These parameter and data is used to estimate acceleration. In this study, two cases are calculated. It is difference how to put external force. These difference are shown each cases.

8.2.1 Observation Data

Actual data is used as the observation data. Observation (Point2) and estimation point (Point1) is shown in Figure.14. Acceleration at estimation point is estimated by the reduced Kalman filter finite element method using observation data. Figure.15,16 and 17 are shown observation of acceleration on X, Y, Z-direction, respectively.

![X-Acceleration at observation point](image)

Figure 15: X-Acceleration at observation point

![Y-Acceleration at observation point](image)

Figure 16: Y-Acceleration at observation point

![Z-Acceleration at observation point](image)

Figure 17: Z-Acceleration at observation point
8.2.2 Blasting Force

When state value is estimated, the way to put external force is the most important thing. The actual blasting at the Ohyorogi tunnel site consists of ten separate blasting. However, the actual observation values using in this study were caused by first blasting force. Therefore, the computation was carried out assuming that the only first blasting was simulated. In this study, concentrated forces are assumed to put on X, Y and Z-direction. Figure.18 and 20 show it. Each time history of external force is shown in Figure.19 and 21.

![Figure 18: X, Y-direction](image1)

![Figure 19: Time history of external force](image2)

![Figure 20: Z-direction](image3)

![Figure 21: Time history of external force](image4)
8.2.3 Results

Figure 22, 23, and 24 are shown comparison of acceleration between observation and estimation data in estimation point. In these figures, a red line is estimation, a blue line is observation.

Figure 22: X-Acceleration at estimation point

Figure 23: Y-Acceleration at estimation point

Figure 24: Z-Acceleration at estimation point

9 CONCLUSION

The reduced Kalman filter finite element method was applied to the model of Ohyorogi tunnel site. Figure 22, 23, and 24 are shown comparison between observation and estimation. There are good in agreement. So, Acceleration could be estimated using actual data.
References


