Ensemble Kalman Filter Finite Element Method
Applied to Dynamic Motion of Elastic Body

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Abstract

The purpose of this research is to obtain an estimation of behavior of an elastic body based on erroneous observations. To do this, the ensemble Kalman filter finite element method is used. In numerical study, a three dimensional elastic body is considered. 2 case studies are carried out. In case 1, cantilever beam is used for the verification. Dynamic motion is caused by the external force applied to the body. As the boundary condition, one side of the beam is fixed in all directions. Observation data of acceleration at observation points are necessity to estimate acceleration at some other points. In this study, artificial data is used. Computed values plus the white Gaussian noise is used as artificial observations. The finite element method is applied to the spatial discretization and the linear acceleration method (Newmark β method) is used as the temporal interpolation. Finally, the estimation is compared with the results by the finite element method, and effectiveness of the EnKF is verified. We suggest that an ensemble of size 5, 10 and 40 to consider the relation between the size of ensemble members and accuracy of the EnKF. In case 2, an actual site is used for the verification. The Iwatayama tunnel is employed. This construction site is located in Gifu prefecture, in Japan. The external force is applied to the tunnel face. As the boundary condition, bottom of the computational model is fixed in all directions. The EnKF is applied to Iwatayama tunnel as well as cantilever beam.

key words : Finite element method, ensemble Kalman filter finite element method, Galerkin method, Newmark β method, cantilever beam, Iwatayama Tunnel

1 Introduction

In civil engineering works, parameter estimation has a close relation to safety. However, research at construction sites requires a lot of time and personal cost. A lot of problems will happen concerning with cost and construction delay. For those reasons, numerical analysis is very important. In recent years, computer technology has made tremendous progress. It is enable us to carry out estimation of a lot of natural phenomena. The finite element method is effective mean of estimation. In general, this method is applied to the numerical simulation in the field of geomechanics. However, in nature, this is not really the case: there is geological uncertainty about different loading conditions, for example. One way to deal with this uncertainty is the random finite element method. Moreover, in order to perform the estimation, the ensemble Kalman filter (EnKF) is adopted. Parameters can be estimated using the limited number of observation data by combining the EnKF with the random finite element method. The present study investigates whether the EnKF finite element method can be adaptable to the dynamic motion of a three dimensional elastic body.
2 The Kalman Filter

2.1 The State Space Model

The Kalman filter is described by two equations. One of them is the system equation;

\[ x_{k+1} = F_k x_k + G_k w_k, \]  

(1)

where \( x_k \) is the state vector at time point \( k \), \( F_k \) is the state transition matrix, \( G_k w_k \) is the system noise, respectively. The other is the observation equation;

\[ y_k = H_k x_k + v_k, \]  

(2)

where \( y_k \) is the observation vector, \( H_k \) is the observation matrix and \( v_k \) is the observation noise, respectively. System noise \( w_k \) is assumed as:

\[ w_k \sim N(0, Q), \]  

(3)

with mean 0 and variance \( Q \). Also observation noise \( v_k \) is assumed as:

\[ v_k \sim N(0, R), \]  

(4)

with mean 0 and variance \( R \). It is supposed that there is no correlation between system and observation noises, that is

\[ E\{w_k, v_k\} = 0, \]  

(5)

where \( E\{\} \) means expectation operator. Observation data are obtained at each time point as:

\[ Y_k = [y_1, y_2, \ldots, y_k]. \]  

(6)

The optimal estimate \( \hat{x}_k \) is an expectation of \( x_k \). Namely observation data \( Y_k \), and \( \hat{x}_k \) is expressed as:

\[ \hat{x}_k = E\{x_k \mid Y_k\}, \]  

(7)

2.2 Kalman Filter Algorithm

The initial condition is given:

\[ \hat{x}_{-1} = \hat{x}_0, \]  

(8)

where \( \hat{x}_0 \) is a known value. The state estimate \( \hat{x}_k \) using the new measurement \( Y_k \) is obtained as follows:

\[ \hat{x}_k = x_k^* + K_k[y_k - H x_k^*], \]  

(9)

where \( K_k \) is called the Kalman gain, and \( x_k^* \) is a priority estimation,

\[ x_k^* = E\{x_k \mid Y_{k-1}\}. \]  

(10)

The estimated error covariance \( P_k \) is expressed as

\[
P_k = \text{cov}\{x_k \mid Y_k\} = E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\} = (I - K_k H)\Gamma_k (I - H K_k^T) + K_k R_k K_k^T \]
\[
= \Gamma_k - K_k H \Gamma_k - \Gamma_k H_k^T K_k^T + K_k (H \Gamma_k H^T + R_k) K_k^T = (I - K_k H)\Gamma_k,
\]  

(11)
where $\Gamma_k$ is called as the predicted error covariance matrix, which is described as

$$
\Gamma_k = \text{cov}\{x_k \mid Y_{k-1}\} = E\{(x_k - x^*_k)(x_k - x^*_k)^T\} = E\{(F_k x_{k-1} + G_{k-1} w_{k-1} - F_k \hat{x}_{k-1}) (F_k x_{k-1} + G_{k-1} w_{k-1} - F_k \hat{x}_{k-1})^T\} = F_k P_{k-1} F_k^T + G_{k-1} Q_{k-1} G_{k-1}^T.
$$  \hfill (12)

Initial condition of the covariance $\Gamma_k$ is:

$$
\Gamma_0 = \hat{\Gamma}_0, \hfill (13)
$$

and $\hat{\Gamma}_0$ is a specified value. The state variable at the next time step is obtained as:

$$
\hat{x}_{k+1} = F_k \hat{x}_k + B. \hfill (14)
$$

At $(k+1)$ time, the estimated error covariance is given by

$$
P_{k+1} = F_k \Gamma_k F_k^T + Q. \hfill (15)
$$

The Kalman gain is derived as

$$
K_k = \Gamma_k H^T (R + H \Gamma_k H^T)^{-1}. \hfill (16)
$$

At $(k+1)$ time, the predicted error covariance can be written as follows.

$$
\Gamma_{k+1} = F_k P_k F_k^T + G_k Q G_k^T. \hfill (17)
$$

3 Finite Element Method

3.1 Basic Equation and Boundary Condition

In this study, indicial notation and the summation convention are used to describe equation. Eqs.(18)—(20) are the basic equations of the elastic body.

(Balance of stress equation)

$$
\sigma_{i,j} - \rho \ddot{u}_i = 0, \hfill (18)
$$

(Strain-displacement equation)

$$
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \hfill (19)
$$

(Stress-strain equation)

$$
\sigma_{ij} = D_{ijkl} \varepsilon_{kl}, \hfill (20)
$$

where $\sigma_{ij}, \rho$ and $\ddot{u}_i$ are overall stress, density and acceleration, respectively. $D_{ijkl}$ is called as the elastic coefficient matrix and described as follows.

$$
D_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \hfill (21)
$$

where $\lambda, \mu$ are Lame’s constants, and $\delta_{ij}$ is Kronecker’s delta. Lame’s constant are described as follows.

$$
\lambda = \frac{\nu E}{(1 - 2\nu)(1 + \nu)}, \hfill (22)
$$

$$
\mu = \frac{E}{2(1 + \nu)}, \hfill (23)
$$

3
where $E$ and $\nu$ are elastic modulus and Poisson ratio, respectively.

The boundaries are divided into the Dirichlet boundary $S_1$ and the Neumann boundary $S_2$. On boundary $S_1$, the displacement is given, and on boundary $S_2$, the surface force is given,

$$u_i = \hat{u}_i \quad \text{on} \quad S_1,$$

$$t_i = \sigma_{ij} n_j = \hat{t}_i \quad \text{on} \quad S_2,$$

where $\hat{u}_i$ and $\hat{t}_i$ are given on the boundary and $n_j$ is normal unit vector.

### 3.2 Finite Element Equation

For the spatial discretization, the finite element method is used. In this study, the Galerkin method is used as an interpolation method. The basic equations are denoted as follows:

$$M_{\alpha i \beta k} \ddot{u}_{\beta k} + K_{\alpha i \beta k} u_{\beta k} = \hat{\Gamma}_{\alpha i},\quad (26)$$

in addition, eq.(26) can be transformed into the following equations considering the damping effect.

$$M_{\alpha i \beta k} \ddot{u}_{\beta k} + C_{\alpha i \beta k} \dot{u}_{\beta k} + K_{\alpha i \beta k} u_{\beta k} = \hat{\Gamma}_{\alpha i},\quad (27)$$

where coefficient matrix are described as follows:

$$M_{\alpha i \beta k} = \rho \int_V \delta_{ik} (N_\alpha N_\beta) dV,\quad (28)$$

$$C_{\alpha i \beta k} = \alpha_0 M_{\alpha i \beta k} + \alpha_1 K_{\alpha i \beta k},\quad (29)$$

$$K_{\alpha i \beta k} = \int_V (N_\alpha, D_{ijkl} N_\beta, l) dV,\quad (30)$$

$$\hat{\Gamma}_{\alpha i} = \int_{S_2} (N_\alpha, \hat{t}_i) dS_2,\quad (31)$$

Here $M_{\alpha i \beta k}$, $C_{\alpha i \beta k}$ and $K_{\alpha i \beta k}$ are mass, damping and stiffness matrix, respectively. $\hat{\Gamma}_{\alpha i}$ is load vector. The linear interpolation function of the finite element method is denoted by $N_\alpha$, and $\alpha_0$ and $\alpha_1$ are both damping coefficients.

### 3.3 The Newmark $\beta$ Method

The Newmark $\beta$ method is applied to the finite element equation as the temporal discretization. At $(n + 1)$ time, eq.(27) can be written as follows;

$$M_{\alpha i \beta k}^{(n+1)} \ddot{u}_{\beta k}^{(n+1)} + C_{\alpha i \beta k} \dot{u}_{\beta k}^{(n+1)} + K_{\alpha i \beta k} u_{\beta k}^{(n+1)} = \hat{\Gamma}_{\alpha i},\quad (32)$$

velocity and displacement at $(n + 1)$ time are expressed as follows;

$$\dot{u}_{\beta k}^{(n+1)} = \dot{u}_{\beta k}^{(n)} + \Delta t \ddot{u}_{\beta k}^{(n)} + \frac{\Delta t}{2} \dddot{u}_{\beta k}^{(n+1)} - \dddot{u}_{\beta k}^{(n)},\quad (33)$$

$$u_{\beta k}^{(n+1)} = u_{\beta k}^{(n)} + \Delta t \dot{u}_{\beta k}^{(n)} + \frac{\Delta t^2}{2} \ddot{u}_{\beta k}^{(n)} + \frac{\Delta t^2}{4} (\dddot{u}_{\beta k}^{(n+1)} - \dddot{u}_{\beta k}^{(n)}),\quad (34)$$

where time increment is $\Delta t$. Using eqs.(33) and (34), eq.(32) can be transformed into:

$$D_{\alpha i \beta k} \ddot{u}_{\beta k}^{(n+1)} = \hat{\Gamma}_{\alpha i} - E_{\alpha i \beta k} \dddot{u}_{\beta k}^{(n)} - J_{\alpha i \beta k} \dddot{u}_{\beta k}^{(n+1)},\quad (35)$$

where each matrix is expressed as follows;

$$D_{\alpha i \beta k} = M_{\alpha i \beta k} + \frac{\Delta t}{2} C_{\alpha i \beta k} + \frac{\Delta t^2}{4} K_{\alpha i \beta k},\quad (36)$$

$$E_{\alpha i \beta k} = \frac{\Delta t}{2} C_{\alpha i \beta k} + \frac{\Delta t^2}{4} K_{\alpha i \beta k},\quad (37)$$

$$J_{\alpha i \beta k} = C_{\alpha i \beta k} + \Delta t K_{\alpha i \beta k}.\quad (38)$$
\( \bar{u}_{j\beta}^{(n+1)} \) is calculated by using eq.(35). Then, these solution are substituted into eqs.(33) and (34), displacement \( \bar{u}_{j\beta}^{(n+1)} \) and \( \dot{\bar{u}}_{j\beta}^{(n+1)} \) can be given.

Eq.(35) is denoted as

\[
D_{\alpha i\beta k} \ddot{u}_k = F_{\alpha i\beta k} \ddot{u}_{n_k} + \psi_{j\beta k}, \tag{39}
\]

\[
F_{\alpha i\beta k} = -\left( \frac{\Delta t}{2} C_{\alpha i\beta k} + \frac{(\Delta t)^2}{4} K_{\alpha i\beta k} \right), \tag{40}
\]

\[
\psi_{j\beta k} = -(J_{\alpha i\beta k} \dot{u}_n + K_{\alpha i\beta k} u_{n_k} - \hat{\Gamma}_{\alpha i}). \tag{41}
\]

Multiplying both sides of eq.(39) and the inverse of \( D_{\alpha i\beta k} \ddot{u}_k \), eq.(39) is the state transition equation mentioned above. Therefore eq.(39) can be considered as the system equation. \( C_{\alpha i\beta k}, K_{\alpha i\beta k}, D_{\alpha i\beta k} \) and \( J_{\alpha i\beta k} \) are expressed in eqs.(29), (30), (36) and (38).

4 The Ensemble Kalman Filter

In the Kalman filter, shown in eq(16), huge computational load is necessary to find the inverse of a matrix. In addition, the errors make prediction of estimation worse. To overcome these difficulties the ensemble Kalman filter is originated by Evensen(1994). An ensemble of possible state vectors are considered, which are generated using the random finite element method, to represent statistical spread of the state vector. The ensemble Kalman filter consists of three steps. The first step is the forecast step: To represent the errorneous behavior, we assume that at time \( k \), an ensemble of \( q \) forecasted state \( x_{f_i}^{(n)} \) estimates random simple error statistics. This ensemble is denoted by \( X_f^k \in \mathbb{R}^{n \times q} \), where

\[
X_f^k \equiv \left( x_{f_1}^k, \cdots, x_{f_q}^k \right). \tag{42}
\]

The ensemble mean \( \bar{x}_f^k \in \mathbb{R}^n \) is defined by

\[
\bar{x}_f^k = \frac{1}{q} \sum_{i=1}^{q} x_{f_i}^k. \tag{43}
\]

An ensemble error matrix \( E_f^k \in \mathbb{R}^{n \times q} \) is expressed as follows.

\[
E_f^k \equiv \begin{bmatrix} x_{f_1}^k - \bar{x}_f^k, \cdots, x_{f_q}^k - \bar{x}_f^k \end{bmatrix}, \tag{44}
\]

and an ensemble set with observation error \( E_a^k \in \mathbb{R}^{n \times q} \) is defined by

\[
E_a^k \equiv \begin{bmatrix} y_{f_1}^k - \bar{y}_f^k, \cdots, y_{f_q}^k - \bar{y}_f^k \end{bmatrix}. \tag{45}
\]

In the ensemble Kalman filter, the forecast state error covariance \( P_{f_k} \) is necessary. The approximate \( P_{f_k}, P_{xy_k} \) and \( P_{yy_k} \) are obtained as:

\[
\hat{P}_{f_k} = \frac{1}{q-1} \hat{E}_{f_k}^T \hat{E}_{f_k}, \tag{46}
\]

\[
\hat{P}_{xy_k} = \frac{1}{q-1} \hat{E}_{f_k}^T \hat{E}_{xy_k}, \tag{47}
\]

\[
\hat{P}_{yy_k} = \frac{1}{q-1} \hat{E}_{yy_k}^T \hat{E}_{yy_k}. \tag{48}
\]

The second step is the analysis step: To obtain the estimate of the state, the EnKF performs an ensemble of the parallel data assimilation cycles, for \( i = 1, \cdots, q \)

\[
x_{a_i}^{(n+1)} = x_{f_i}^{(n+1)} + k \left( y_{f_i}^{(n+1)} - h \left( x_{f_i}^{(n)} \right) \right). \tag{50}
\]
The perturbed observations $y_i^k$ are assumed as:

$$y_i^k = y_k + v_i^k.$$  

(51)

Thinking of the forecasted ensemble mean as the best estimate of the state, and spread of the ensemble members as the error between best estimate and true state, we approximate the analysis error covariance as follows:

$$\hat{P}_k^a \equiv \frac{1}{q-1} E_k^a \left( E_k^a \right)^T,$$  

(52)

where $E_k^a$ is defined by Eq.(45), and $x_{fi}^k$ replaced by the mean of the analysis estimate ensemble members. Then, the Kalman gain is obtained by the approximation of the error covariance as,

$$K_k = \hat{P}_{xy}^f \left( \hat{P}_{yy}^f \right)^{-1}.$$  

(53)

In the EnKF, the Kalman gain can be obtained using the square matrix of order $q$.

The last step is to obtain the prediction of error statistics using the values of the forecast step:

$$x_{k+1}^{fi} = f(x_k^{ai}, u_k) + w_k^i.$$  

(54)

5 Numerical Study 1

5.1 A Three Dimensional Elastic Body: Cantilever Beam

In this case study, cantilever beam is used for verification. Acceleration of elastic body is estimated by using ensemble in different size. The three dimensional elastic body for the use of verification is shown in Figure 22. The total number of nodes and elements are 1029 and 4320, respectively. The size of the body is $60 \times 200 \times 60[m]$. As a boundary condition, on one side of the body is fixed, and the other side is free. Poisson ratio and density are set $0.30$ and $2 \times 10^3[kg/m^3]$, respectively. Each element is assumed to have a random elastic modulus. In this study damping coefficient $\alpha_0$ and $\alpha_1$ are set $0.001[1/sec]$ and $0.01[m/sec]$.

The beam is subjected to the external force, which is expressed by

$$\hat{\Gamma}_{ai} = \int_{\Gamma} A_{ai} \left( e^{-\xi t} - e^{-\eta t} \right) dS,$$  

(55)
where $A_{oi}$ is the magnitude of external force. The parameter $A_{oi}$ is randomized.

Figure 2 shows a time history of external force. A time increment $\Delta t$ is 0.05[s]. The random finite element method combines the finite element method with the random field theory. In this study, the elastic modulus and magnitude of the external force are randomized. Elastic modulus based on a standard distribution can be transformed into those on a normal distribution. For each element:

$$E_i = \mu_E + \sigma_E Z_i,$$

(56)

where $\mu_E$ is mean of the elastic modulus, and $\sigma_E$ is standard deviation of the elastic modulus, and $i$ means number of element. $\mu_E$ is set $1.0 \times 10^7[kN/m^3]$ in this study.

$$A_{oi} = \mu_{A_{oi}} + \sigma_{A_{oi}} Z_i,$$

(57)

where $\mu_{A_{oi}}$ is mean of magnitude of the external force, and $\sigma_{A_{oi}}$ is the standard deviation of the magnitude. $\mu_{A_{oi}}$ is set $1.0 \times 10^7[kN/m^3]$.

The random variable based on standard normal distribution is represented by $Z_i$.

Observation and estimation points are set as shown in Figure 23. Acceleration at estimation point is estimated using the observation data. This data is shown in the next section.
5.2 Observation Data and Ensemble Members

All components of the acceleration at the observation point are illustrated in Figures. Observation data are obtained by the random finite element method. 5, 10, and 40 ensemble members are used at each time step. In these figures, a solid line is ensemble mean.

The distribution of the ensemble is shown in Figures 4, 5 and 6 when 5 ensemble members are used. These figures show values on X,Y,Z-direction, respectively.

Figure 4: the ensemble at X-direction

Figure 5: the ensemble at Y-direction

Figure 6: the ensemble at Z-direction
The spread of the ensemble is shown in Figures 7, 8 and 9 when 10 ensemble members are used.

Figure 7: the ensemble at X-direction

Figure 8: the ensemble at Y-direction

Figure 9: the ensemble at Z-direction
The spread of the ensemble is shown in Figures 10, 11 and 12 when 40 ensemble members are used.

Figure 10: the ensemble at X-direction

Figure 11: the ensemble at Y-direction

Figure 12: the ensemble at Z-direction
5.3 Verification

Figures 30, 31 and 32 show the comparisons of acceleration between estimation value using the EnKF finite element method and artificial observation value using the finite element method. The estimation values can be found between the range of the ensemble members. A low number ensemble members make the range of error large. In contrast, if the number of ensemble members is increased accuracy of the EnKF is leveled up. It is seen that the estimated acceleration with 40 ensemble members is almost coincident with the observed acceleration. The filtering performed well.

Figure 13: X-acceleration at estimation point

Figure 14: Y-acceleration at estimation point

Figure 15: Z-acceleration at estimation point
To obtain better accuracy, we should determine the number of the ensemble members. We start with comparing the size of the ensemble members. We compare them next.

Figure 16: X-acceleration at estimation point

Figure 17: Y-acceleration at estimation point

Figure 18: Z-acceleration at estimation point
6 Numerical Study 2

6.1 A Three Dimensional Elastic Body: Iwatayama Tunnel

Next study, state value is estimated using an actual site. We employ a construction site of Iwatayama tunnel which is located in Gifu prefecture, Japan. Project to build a tunnel place of Mt.Iwata is carried out. Iwatayama tunnel is, at 1001[m] in length, two-lane bypass. The mountain is close to the residential area, and has rock slope. Therefore it is important that make a close examination of rock vibration.

![Figure 19: aerial view of Mt.Iwata](image1)

Figure 19: aerial view of Mt.Iwata

![Figure 20: the tunnel mouth of Iwatayama tunnel](image2)

Figure 20: the tunnel mouth of Iwatayama tunnel
The elastic body for the use of estimation is shown in Figure 22. The total number of nodes and elements are 3746 and 18678, respectively. The size of the body is $200 \times 200 \times 110\,[m]$. As a boundary condition, bottom is fixed, and the other surface is free. Poisson ratio and density are set 0.30 and $2.0 \times 10^3\,[kg/m^3]$, respectively. Each element is assumed to have a random elastic modulus. In this study damping coefficient $\alpha_0$ and $\alpha_1$ are set $0.001[1/sec]$ and $0.01[m/sec]$. 
In this case, external force is expressed same as case 1.

\[ \hat{\Gamma}_{ai} = \int_{\Gamma} A_{ai} \left( e^{-\xi t} - e^{-\eta t} \right) dS. \] (58)

The elastic modulus and magnitude of the external force are randomized. \( E \) is set \( 1.0 \times 10^{7}[\text{kN/m}^3] \), and \( A_{ei} \) is set \( 1.0 \times 10^{7}[\text{kN/m}^3] \).

Observation and estimation points are set as shown in Figure 23. Acceleration at estimation point is estimated using the observation data. This data is shown in the next section.

![Figure 23: observation point and estimation point](image-url)
6.2 Observation Data and Ensemble Members

All components of the acceleration at the observation point are illustrated in Figures 24, 25 and 26. These figures show values on X,Y,Z-direction, respectively. Observation data are obtained by the random finite element method. The distribution of the ensemble is shown in Figures 24, 25 and 26. Ten ensemble members are used at each time step.

Figure 24: the ensemble at X-direction

Figure 25: the ensemble at Y-direction

Figure 26: the ensemble at Z-direction
6.3 Verification

Figures 30, 31 and 32 show the comparisons of acceleration between estimation value using the EnKF finite element method and artificial observation value using the finite element method. The estimation values can be found between the range of the ensemble members. It is seen that the estimated acceleration is almost coincident with the observed acceleration. The filtering performed well.

Figure 27: X-acceleration at estimation point

Figure 28: Y-acceleration at estimation point

Figure 29: Z-acceleration at estimation point
We make comparison between results with 10 ensemble members and 5 ensemble members. The estimation data of 10 members accords with observation data besides of 5 members.

Figure 30: X-acceleration at estimation point

Figure 31: Y-acceleration at estimation point

Figure 32: Z-acceleration at estimation point
7 Conclusion

It is found that the acceleration at estimation point can be estimated. The EnKF finite element method can be successfully applied to the cantilever beam and the Iwatayama tunnel model. The results of numerical experiments show that the estimated data of acceleration are well in agreement with the observed data. It is observed that, the accuracy of the EnKF increases when the number of ensemble members grows. As the future work, state value is estimated using actual measured data. In addition, we should increase the number of ensemble members.

References


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