Minimization of Lift Force of Body Located in Viscous Flows Using Adjoint Equation Method

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Abstract

The purpose of this study is to determine an angle of a body located in an incompressible viscous flow minimizing the lift force of the body. At present, there have been some results on the optimal shape of a body. The optimal shape is defined so as to minimize fluid forces action. Optimization is carried out by the combined methods of the finite element method and optimal control theory, in which a performance function is expressed by the fluid force. In this study, we find the optimal angle of attack in an incompressible viscous flow by applying this method. An optimal angle of attack is defined by minimizing the lift force to act on body. In the future, it will take an important part in the construction of the bridge. In order to minimize the lift force on a body, the performance function is introduced. The performance function is defined by the square sum of the fluid forces of a body. This problem can be transformed into the non-constrained minimization problem by the Lagrange multiplier method, then we can obtain the extended performance function. The adjoint equations can be derived by the stationary condition of variation of the extended performance function. We can derive the gradient to update the angle of the body from solving the adjoint and the state equations. As a minimization technique, the weighted gradient method based on the mixed interpolation is applied. We employ the Navier-Stokes equation as the state equation. The Galerkin finite element method is applied to the state equation as spatial discretization, and the Crank-Nicolson method is applied as the temporal discretization. As numerical studies, the angle of body located in the flow is carried out at Reynolds number 100.

Key Words: Performance function, adjoint equation, angle of attack, The Galerkin finite element method

1 INTRODUCTION

In the past, reasonable shape of a body located in the flow is obtained by experiments, which need a lot of cost and time. Recently, the computational fluid dynamics can be used according to the advancement of numerical techniques and computer hardwares. Fluid flow over the cylinder can be easily computed by the numerical analysis. The optimal shape can be obtained by applying numerical analysis and optimization theory. Moreover, the determination of shape optimization of a body is excessively advanced.

In this study, we would find an angle of attack of a body located in the fluid flow. Most of fluid flows in engineering field are considered as incompressible. The condition of this study shows as; 1) The optimal control is considered to be equal to minimization of the fluid force applied to the object. 2) For the fluid force, only the lift force is considered. 3) The drag force is disregarded in this research because we suppose that destruction of oscillation. The adjoint method is used for the optimization technique. The adjoint method is used as a minimization technique of the lift force. The performance function of the lift force is employed and it is made minimum value. The first variation is taken from the extended performance function. The adjoint equations can be derived and the inverse analysis is carried out. The performance function is an index to evaluate that the function is minimized in the
adjoint method. The angle of a body is updated using the gradient. The gradient is obtained by solving the adjoint
equation. In the case that the angle of the body is updated, also it is necessary to update the mesh. To do this,
it is necessary to change the coordinate system of the gradient into the polar coordinate system. The angle of the
body obtained by these operations.

As the discretization technique, the finite element method is used to the spatial discretization and the Crank-
Nicolson method is applied to the temporal discretization. In this study, a prism shape in the fluids are analyzed.
The computational domain is two dimensional space filled with the incompressible fluid.

2 BASIC EQUATION

Let two dimensional space be denoted by $\Omega$ and $\Gamma$ be the boundary of $\Omega$. The flow field is assumed to be an
incompressible viscous fluid. The basic equation can be derived as follows;

$$
\dot{u}_i + u_j u_{i,j} + p_i - \nu(u_{i,j} + u_{j,i})_j = 0 \quad \text{in} \quad \Omega,
$$

(1)

$$
u u_{i,i} = 0 \quad \text{in} \quad \Omega,
$$

(2)

where, $u_i, p, \nu$ are the velocity, pressure and kinematic viscosity coefficient, respectively. The computational domain
is shown as follows:

![Figure1: Computational domain and boundary condition](image)

The boundary $\Gamma$ consists of inflow $\Gamma_U$, edge $\Gamma_S$, outflow $\Gamma_D$, and body $\Gamma_B$ boundaries, respectively. The boundary
conditions are given as follows:

$$
u_i = \dot{u}_i \quad \text{on} \quad \Gamma_U,
$$

(3)

$$u_2 = 0, t_1 = 0 \quad \text{on} \quad \Gamma_S,
$$

(4)

$$t_i = 0 \quad \text{on} \quad \Gamma_D,
$$

(5)

$$u_i = 0 \quad \text{on} \quad \Gamma_B,
$$

(6)

$$t_i = [-p\delta_{i,j} + \nu(u_{i,j} + u_{j,i})]n_j.
$$

(7)

where $\dot{u}_i, t_i$ and $n_j$ are already known inflow velocity, traction vector and unit outward normal to the boundary
in $\Gamma$. 

2
3 DISCRETIZATION

3.1 Spatial discretization

For the spatial discretization, the finite element method presented by Matsumoto et al. [1999][2000] is applied. The bubble function element is applied to velocity interpolation, and the linear element is applied to pressure interpolation.

\[
\begin{align*}
    u_i &= \Phi_1 u_{i1} + \Phi_2 u_{i2} + \Phi_3 u_{i3} + \Phi_4 u_{i4} = \Phi_i u_{i3}, \quad (8) \\
    \bar{u}_{i4} &= u_{i4} - \frac{1}{3}(u_{i1} + u_{i2} + u_{i3}), \quad (9) \\
    \Phi_1 &= l_1, \Phi_2 = l_2, \Phi_3 = l_3, \Phi_4 = 27l_1l_2l_3 \quad (10)
\end{align*}
\]

\[
\begin{align*}
    p &= \Psi_1 p_1 + \Psi_2 p_2 + \Psi_3 p_3 = \Psi_i p_i, \quad (11) \\
    \Psi_1 &= l_1, \Psi_2 = l_2, \Psi_3 = l_3, \quad (12)
\end{align*}
\]

in which \(l_1l_2l_3\) means area coordinate.

\[\text{Figure 2: bubble function element} \quad \text{Figure 3: linear function element}\]

The finite element equation can be expressed as follows:

\[
\begin{align*}
    M_{\alpha\beta} \ddot{u}_{\beta1} + A_{\alpha\beta\gamma} u_{\beta j} u_{\gamma1} - H_{\alpha1\lambda} p_\lambda + D_{\alpha\beta\lambda} u_{\beta j} &= T_{\alpha i}, \quad (13) \\
    H_{\lambda\alpha i} u_{\alpha i} &= 0. \quad (14)
\end{align*}
\]

3.2 Temporal discretization

As for the time discretization, the Crank-Nicolson method is applied to the finite element equation.

\[
\begin{align*}
    M_{\alpha\beta} \frac{u_{\beta1}^{n+1} - u_{\beta i}^n}{\Delta t} + A_{\alpha\beta\gamma} u_{\beta j} u_{\gamma1}^{n+\frac{1}{2}} - H_{\alpha1\lambda} p_\lambda^{n+1} + D_{\alpha\beta\lambda} u_{\beta j}^{n+\frac{1}{2}} &= T_{\alpha i}, \quad (15) \\
    H_{\lambda\alpha i} u_{\alpha i}^{n+1} &= 0. \quad (16)
\end{align*}
\]
4 FORMULATION

4.1 Performance function

In this study, the minimization problem of the fluid force acting on the body with respect to the angle of attack is treated. The performance function $J$ is defined by the square sum of the fluid forces as follows.

$$J = \frac{1}{2} \int_{t_0}^{t_f} (q_{\alpha i \beta j} F_{\alpha i} F_{\beta j}) dt,$$

where $F_{\alpha i}$ means fluid force applied to the body and $q_{\alpha i \beta j}$ are the weights, respectively. In order to obtain the optimal angle of attack, the performance function defined by the fluid force should be minimized. The performance function should be minimized satisfying the constraint conditions which are eqs (1) and (2). The Lagrange multiplier for eqs (1) and (2) are defined by adjoint velocity $u_i^*$ and pressure $p_j^*$. The performance function is extended to $J^*$ as follows:

$$J^* = \frac{1}{2} \int_{t_0}^{t_f} (q_{\alpha i \beta j} F_{\alpha i} F_{\beta j}) dt$$

$$+ \int_{t_0}^{t_f} u_i^* (M_{\alpha i \beta j} u_j^* + A_{\alpha i \gamma j} u_{\gamma j} - H_{\alpha i \lambda} p_\lambda + D_{\alpha i \beta j} u_j^* - T_{\alpha i}) dt$$

$$+ \int_{t_0}^{t_f} p_j^* H_{\beta j \eta} u_{\beta j} dt.$$

(18)

$J^*$ is referred to as the extended performance function.

4.2 First order adjoint equation

The minimization problem with constraint conditions results in solving the stationary conditions of the extended performance function $J^*$. The first order adjoint equation is expressed as follows.

$$\delta J^* = \int_{t_0}^{t_f} \delta u_i^* (M_{\alpha i \beta j} u_j^* + A_{\alpha i \gamma j} u_{\gamma j} - H_{\alpha i \lambda} p_\lambda + D_{\alpha i \beta j} u_j^* - T_{\alpha i}) dt$$

$$+ \int_{t_0}^{t_f} \delta p_j^* H_{\beta j \eta} u_{\beta j} dt$$

$$+ \int_{t_0}^{t_f} \delta u_j (M_{\alpha i \beta j} u_{\alpha i} + B_{\alpha i \beta j} u_{\alpha i} - H_{\beta j \eta} p_\eta^* + D_{\alpha i \beta j} u_{\alpha i}) dt$$

$$+ \int_{t_0}^{t_f} \delta p_\lambda H_{\alpha i \lambda} u_{\alpha i} dt$$

$$- \int_{t_0}^{t_f} \delta T_{\alpha i} (u_{\alpha i}^* - q_{\alpha i \beta j} F_{\beta j}) dt$$

$$+ G_{\delta k} \frac{\partial X_{s k}}{\partial \varphi} \delta \varphi.$$

(19)

where

$$B_{\alpha i \beta j} = A_{\alpha \beta \gamma j} u_{\beta j} + A_{\alpha \gamma \beta k} u_{\gamma k}$$

(20)

The stationary condition means that the first variation of the extended performance function is equal to zero. Therefore, the first order adjoint equations are obtained as follows:
\( M_{\alpha\beta\gamma} \dot{u}_{\beta\gamma} + A_{\alpha\beta\gamma} u_{\gamma\beta} - H_{\alpha\gamma} p_{\alpha} + D_{\alpha\beta\gamma} u_{\beta\gamma} = T_{\alpha}, \quad \text{in } \Omega, \) \hspace{1cm} (21)

\( H_{\beta\gamma} u_{\beta\gamma} = 0 \quad \text{in } \Omega, \) \hspace{1cm} (22)

\(-M_{\alpha\beta\gamma} \dot{u}^*_{\alpha\gamma} + B_{\alpha\beta\gamma} u^*_{\alpha\gamma} - H_{\beta\gamma} p^*_{\gamma} + D_{\alpha\beta\gamma} u^*_{\alpha\gamma} = 0 \quad \text{in } \Omega, \) \hspace{1cm} (23)

\( H_{\alpha\beta\gamma} u^*_{\alpha\gamma} = 0 \quad \text{in } \Omega, \) \hspace{1cm} (24)

\( u^*_{\alpha\gamma} - q_{\gamma\alpha\beta} F_{\beta\gamma} = 0 \quad \text{on } \Gamma_B, \) \hspace{1cm} (25)

\( G_{\delta k} \frac{\partial X_{\delta k}}{\partial \varphi} = 0 \quad \text{in } \Omega. \) \hspace{1cm} (26)

The gradient of \( J^* \) with respect to \( \varphi \) can be derived as;

\[
\text{grad}(J^*) = G_{\delta k} \frac{\partial X_{\delta k}}{\partial \varphi},
\] \hspace{1cm} (27)

where \( X_{\delta k} \) is coordinates of the surface of a body, and \( G_{\delta k} \) are shown as follows,

\[
G_{\delta k} = \int_{t^0}^{t_f} \left( u^*_{\alpha\gamma} \frac{\partial M_{\alpha\beta\gamma}}{\partial X_{\delta k}} \dot{u}_{\beta\gamma} + u^*_{\alpha\gamma} \frac{\partial X_{\delta k}}{\partial X_{\delta k}} u_{\gamma\beta} - u^*_{\alpha\gamma} \frac{\partial H_{\alpha\gamma}}{\partial X_{\delta k}} p_{\alpha} + u^*_{\alpha\gamma} \frac{\partial D_{\alpha\beta\gamma}}{\partial X_{\delta k}} u_{\beta\gamma} + p^*_{\gamma} \frac{\partial H_{\beta\gamma}}{\partial X_{\delta k}} u_{\beta\gamma} \right) dt. \] \hspace{1cm} (28)

5 MINIMIZATION

5.1 Weighted gradient method

As the minimization technique, the weighted gradient method is applied. A modified performance function \( K \) can be obtained by adding a penalty term to the extended performance function, which is expressed as follows,

\[
K^{(l)} = J^{(l)} + \frac{1}{2} \int_{\Gamma_n} (\varphi_i^{(l+1)} - \varphi_i^{(l)}) W_{ij}(\varphi_j^{(l+1)} - \varphi_j^{(l)}) d\Gamma,
\] \hspace{1cm} (29)

where \( l \) and \( W_{ij} \) are iteration count and the stabilizing weight, respectively. In case that the modified performance function \( K \) converges to the minimum, the penalty term can be zero. To minimize the modified performance function \( K \) is equal to minimize the extended performance function \( J^* \). If the following stationary condition is applied to the modified performance function,

\[
\delta K^{(l)} = 0 \), \hspace{1cm} (30)

then, the update of the angle of attack is calculated at each iteration cycle by the following equation:

\[
W_{\varphi}^{(l+1)} = W_{\varphi}^{(l)} - G_{\delta k} \frac{\partial X_{\delta k}}{\partial \varphi}. \] \hspace{1cm} (31)
6 NUMERICAL STUDY

The computational domain and boundary condition are shown in Figure 4. The weighting parameter $q_2$ is 1.0. The time increment is set to 0.01.

The finite element mesh is represented in Figure 5. The total number of nodes and elements are 8627 and 17044, respectively. The surface of the body consists of 120 nodes. The problem is to find the optimal angle in viscous flow of Reynolds number 1.

Case 1: symmetric
Case 2: asymmetric

figure 5: Computational domain and boundary condition (case 1)

figure 6: Finite element mesh (case 1)
7 NUMERICAL RESULTS
As the numerical results, the computed variables which are obtained by the optimal control are shown. The variation of the performance function is shown in Figure.6. The variation of the angle of attack is shown in Figure.7.
figure 9: Performance function (case1)

figure 10: The variation of angle of wing (case1)
Figure 11: Counter of initial angle (case 1)

Figure 12: Counter of final angle (case 1)
figure 13 : Performance function (case2)

figure 14 : The variation of angle of wing (case2)
figure 15: counter of initial angle (case 2)

figure 16: counter of final angle (case 2)
8 CONCLUSION

In this study, a minimization of lift force of a body located in an incompressible flow is presented. The lift force has been reduced by finding the angle of attack optimized. As the future work, we should make analysis on the condition of the higher Reynolds number.

References


