Optimal Control of Shallow Water Flows
Using Adjoint Equation Method

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Abstract

This paper presents a method to control a flow behavior using the first-order adjoint equation method. A flood causes large-scale and extensive damage to the human property. It is expected that the damage can be suppressed to minimum if the water level or water discharge of the river can be controlled. Therefore, the optimal control of a water flow is carried out in this study. In the control theory, as the index of the optimal control, the performance function which is defined by the square sum of the computed water elevation at the target points and the square sum of the control value is used. The extended performance function is given by the performance function and the state equation. The first-order adjoint equation can be derived by the condition that the first variations of the extended performance function and constraint condition are zero. The gradient of the extended performance function is obtained by solving the first-order adjoint equation. As the minimization technique, the weighted gradient method is applied. The shallow water equation based on the water discharge and elevation is used as a state equation. As the spatial discretization, the linear function interpolation is employed. As the temporal discretization, the two-steps explicit scheme is applied. As the verification, the optimal control of water elevation in simple rectangular channel model is performed. In numerical studies, the optimal control of water elevation in the Ikari dam lake is carried out. There is the Kawaji dam close to the Ikari dam, and the Ikari dam lake has the aqueduct with pump to send the water to the Kawaji dam lake. In this research, the optimal water discharge sent by the pump to minimize the water rise in the Ikari dam lake is computed. The inflow boundary conditions are presupposed as sinusoidal wave water elevation. The results of optimal control is performed effectively.

Keywords: Finite Element Method, Shallow Water Flow, Two-steps explicit scheme, Performance Function, First-order Adjoint Equation, Weighted Gradient Method, Ikari Dam

1 Introduction

Recently, the flood damage becomes one of the severe social problem all over the world. A flood damage causes a large scale personal sufferings and economical damages. Although the bank is constructed to reduce such damage, it is still difficult to forecast such floods. If the flood exceeds the plan volume of the water, the excessive water flows out the outside of river. Alternatively, dams are constructed to prevent water flows to the downstream. However,
it is sometimes too late to discharge the overflow water in case that sudden heavy rain is occurred. They causes the dam break and the risk of downstream by a large amount of overflow. In order to prevent such damage, various preventions should be made.

For example, Kinu river system located in Tochigi prefecture in Japan has a number of dams, and has network between the Kawaji dam lake and the Ikari dam lake. The Kawaji dam has a larger available storage capacity than the Ikari dam has. In case that the flood that exceeds the acceptable quantity of a dam lake occur, the water is sent to the other dam lake running through the water tunnel. Herewith, it is possible to prevent the dam break and to reduce quantity of discharge to the downstream. The optimal control of the intake water has not been carried out yet. It is necessary to control the intake water discharge to keep the downstream safe.

Consequently, in this research, the verification to control the water elevation at target point in the river is carried out. Specifying discharge at the boundary in the computational domain, the optimal control of the water elevation to be as small as possible is carried out by flowing water discharge from the control boundary. The control variable should be computed so as to minimize the performance function under the constraint of the state equation and boundary condition. The performance function is defined by the square sum of difference between the computed and the object value of the water elevations at the target points. The minimization of the performance function means that computed value becomes as close as possible to the object value. The performance function is extended by adding inner products between the Lagrange multipliers and the state equations. The Lagrange multiplier method is suitable for the minimization problem with constraint condition. The first-order adjoint equation can be derived by the state equation and the boundary conditions. As the minimization technique, the weighted gradient method is applied, which is used widely for the minimization technique. Control variable is updated by the gradient which is obtained by solving the state and adjoint equations. The conservative shallow water equation are employed as the state equation. The river and the lake are taken as a computational domain. As the discretization technique, the finite element method using the linear function interporation in the space direction and the two-steps explicit scheme in time are applied.

2 State Equation

The non linear shallow water equations are employed as a state equation, which are expressed as follows using indicial notation and summation convention:

\[ \dot{q}_i + (u_j q_i)_j + g(\eta + H)\eta_i = 0 \]
\[ \dot{\eta} + q_i = 0, \]

where \( q_i \), \( u_i \), \( g \), \( \eta \) and \( H \) are water discharge, water velocity, gravitational acceleration, water elevation and water depth, respectively. The boundary conditions shown in Figure 2 are given as follows:

\[ q_i = \hat{q}_i \quad \text{on} \quad \Gamma_D, \]
\[ Q = q_i u_i = \hat{Q} \quad \text{on} \quad \Gamma_S, \]
\[ q_i = U_i \quad \text{on} \quad \Gamma_C, \]

where the boundaries \( \Gamma_D, \Gamma_S, \) and \( \Gamma_C \) are the Dirichlet, slip, and control boundaries, respectively and \( U_i \) is control variable.

The initial conditions are given as follows:

\[ q_i(t_0) = \hat{q}_i \quad \text{in} \quad \Omega, \]
\[ \eta(t_0) = \hat{\eta} \quad \text{in} \quad \Omega. \]
3 Discretization Technique

As the discretization technique, the finite element method using linear function interpolation is applied in spatial discretization, and the two step explicit scheme is applied in temporal discretization.

3.1 Spatial Discretization

As for the spatial discretization of the state equations based on discharge and water elevation, the linear function interpolation which is shown in Figure 3 is applied.

\[
q_i = \Phi_1 q_{\beta_1} + \Phi_2 q_{\beta_2} + \Phi_3 q_{\beta_3}, \quad (8)
\]

\[
\eta = \Phi_1 \eta_1 + \Phi_2 \eta_2 + \Phi_3 \eta_3, \quad (9)
\]

\[
\Phi_1 = L_1, \quad \Phi_2 = L_2, \quad \Phi_3 = L_3. \quad (10)
\]

From these discretization technique, the finite element equations of the state equations are expressed as follows:

\[
M_{\alpha\beta}\dot{q}_{\beta i} + A_{\alpha\beta\gamma i}q_{\beta i}u_{\gamma i} + A_{\alpha\beta\gamma j}u_{\gamma j}q_{\beta i} + A_{\alpha\beta\gamma}\eta_{\beta}(\eta_{\gamma} + H_{\gamma}) + B_{\alpha i\beta j}(\eta_{\beta} + H_{\beta})\eta_{\gamma} = 0 \quad (11)
\]

\[
M_{\alpha\beta}\dot{\eta}_{\beta i} + S_{\alpha i\beta}\dot{q}_{\beta i} = 0 \quad (12)
\]

where,

\[
M_{\alpha\beta} = \int_{\Omega}(\Phi_\alpha \Phi_\beta)d\Omega, \quad (13)
\]

\[
A_{\alpha\beta\gamma i} = \int_{\Omega}(\Phi_\alpha \Phi_\beta \Phi_{\gamma i})d\Omega, \quad (14)
\]

\[
B_{\alpha i\beta j} = \int_{\Omega}(\Phi_\alpha \Phi_{\beta j} \Phi_{\gamma})d\Omega, \quad (15)
\]

\[
S_{\alpha\beta i} = \int_{\Omega}(\Phi_\alpha \Phi_{\beta i})d\Omega. \quad (16)
\]
3.2 Temporal Discretization

As for the temporal discretization of the state equations, the two-step explicit scheme is applied: Considering the Taylor expansion of a function $f(t)$.

$$f(t + \Delta t) = f(t) + \Delta \dot{f}(t) + \frac{\Delta t^2}{2} \ddot{f}(t) + \ldots$$

(17)

where $\Delta t$ is a time increment. Equation (17) can be divided by employing the second term into the following two equations:

$$f\left(t + \frac{\Delta t}{2}\right) = f(t) + \frac{\Delta t}{2} \dot{f}(t)$$

(18)

$$f(t + \Delta t) = f(t) + \Delta t \dot{f}\left(t + \frac{\Delta t}{2}\right)$$

(19)

By using this technique, the finite element equation can be transformed by explicit solution method. Applying equation (18) and (19) to equation (11) and (12), the finite element equation using two-step explicit scheme is obtained as following equation:

First step:

$$M_{\alpha\beta} q_{\beta i}^{n+\frac{1}{2}} = \tilde{M}_{\alpha\beta} q_{\beta i}^{n} - \frac{\Delta t}{2} \left\{ A_{\alpha\beta\gamma j} q_{\gamma j}^{n} u_{\gamma j}^{n} + A_{\alpha\beta\gamma j} u_{\beta j}^{n} q_{\gamma j}^{n} + A_{\alpha\beta j} \eta_{\beta j}^{n} (\eta_{\gamma i}^{n} + H_{\gamma}) + B_{\alpha\beta j} (\eta_{\beta j}^{n} + H_{\beta}) \eta_{\beta j}^{n} \right\}$$

(20)

Second step:

$$M_{\alpha\beta} \dot{q}_{\beta i}^{n+\frac{1}{2}} = \tilde{M}_{\alpha\beta} q_{\beta i}^{n} - \Delta t \left\{ + A_{\alpha\beta\gamma j} q_{\gamma j}^{n+\frac{1}{2}} u_{\gamma j}^{n+\frac{1}{2}} + A_{\alpha\beta j} u_{\beta j}^{n+\frac{1}{2}} q_{\gamma j}^{n+\frac{1}{2}} \right\}$$
\[ + A_{\alpha\beta\gamma} \eta_{\beta}^{n+\frac{1}{2}} (\eta_{\gamma}^{n+\frac{1}{2}} + H_{\gamma}) + B_{\alpha i\beta\gamma} (\eta_{\beta}^{n+\frac{1}{2}} + H_{\beta}) \eta_{\gamma}^{n+\frac{1}{2}} \}
\]

\[ M_{\alpha\beta} \eta_{\beta}^{n+1} = \bar{M}_{\alpha\beta} \eta_{\beta}^{n} - \Delta t S_{\alpha\beta} \eta_{\beta}^{n+\frac{1}{2}} \]

where, \( \bar{M}_{\alpha\beta} \) is centralized matrix as follows;

\[ \bar{M}_{\alpha\beta} = \begin{bmatrix} \bar{m} & \bar{m} & \bar{m} \\ \bar{m} & \bar{m} & \bar{m} \end{bmatrix} = A_{e} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \]

In this case, the left part factor is the matrix composed of only diagonal elements. Thus, calculation of the inverse matrix is the calculation of the reciprocal of the diagonal elements. However, it is known when \( \bar{M} \) is employed as the first term of right part, the calculation becomes numerical instability. Whereat, in order to solution this problem, following coefficient matrix is employed as the first term of the right part;

\[ \bar{M}_{\alpha\beta} = e M_{\alpha\beta} + (1 - e) M_{\alpha\beta}, \]

where, \( e \) is lumping parameter to introduce an artificial viscosity and firm up the computation. The value of this lumping parameter \( e \) is changed by the problems.
4 Optimal Control

4.1 Performance Function

In this paper, the control problem is defined to find the control discharge so as to minimize the performance function under the constraints of the state equation. The performance function $J$ is defined by the square sum of the discrepancy between the computed and the object value of the water elevations at the target points. The minimization of the performance function $J$ means that the computed water elevation at the target point becomes as close as possible to the objective water elevation.

$$J = \frac{1}{2} \int_{t_0}^{t_f} \int_{\Omega} (\eta - \eta_{obj})Q(\eta - \eta_{obj})d\Omega dt,$$

where $Q, \eta$ and $\eta_{obj}$ are weighting diagonal matrix, the computed and the objective water elevation, respectively.

4.2 First-order Adjoint Equation

The Lagrange multiplier method is suitable for the minimization problem with constraint conditions. The performance function is extended by adding inner products between Lagrange multipliers and the state equations. The extended performance function $J^*$ is expressed as follows;

$$J^* = J + \int_{t_0}^{t_f} \int_{\Omega} q_i^* \left( \dot{q}_i + (u_j q_i)_j + g(\eta + H)\eta_i \right) d\Omega dt$$

$$+ \int_{t_0}^{t_f} \int_{\Omega} \eta^* \left( \dot{\eta} + q_i \right) d\Omega dt,$$

where $q_i^*$ and $\eta^*$ denote the Lagrange multiplier for water discharge and water elevation, respectively. Where, the water velocity $u_i$ is able to be substituted as follows:

$$u_i = \frac{q_i}{(\eta + H)}.$$

The first variation of the extended performance function $\delta J^*$ is expressed as follows.

$$\delta J^* = \int_{t_0}^{t_f} \int_{\Omega} \delta q_i \left( -\dot{q}_i - u_j q_i^*_j - \eta_i^* - \frac{1}{\xi} q_j q_j^*_i \right) d\Omega dt$$

$$+ \int_{t_0}^{t_f} \int_{\Omega} \delta \eta \left( -\dot{\eta} - g(\eta + H)q_i^* + g\eta_i^* H_i + \frac{1}{\xi} q_j q_j^* \eta_i + Q(\eta - \eta_{obj}) \right) d\Omega dt$$

$$+ \int_{t_0}^{t_f} \int_{\Gamma} \delta q_i \left( \eta^* \eta_i + \frac{1}{\xi} q_j q_j^* + u_j q^*_i n_j \right) d\Gamma dt$$

$$+ \int_{t_0}^{t_f} \int_{\Gamma} \delta \eta \left( (\eta + H)q_i^* - \frac{1}{\xi} q_j q_j^* \eta_i \right) n_i d\Gamma dt.$$

The stationary condition that is the first variation of $J^*$ should be zero is given to minimize the performance function.

$$\delta J^* = 0.$$

Then, the first-order adjoint equation eq.(31) , (32), the terminal conditions eq.(33), (34), and the boundary conditions eq.(35) can be obtained:

$$-\dot{q}_i^* - u_j q_i^*_j - \eta_i^* - \frac{1}{\xi} q_j q_j^*_i = 0 \text{ in } \Omega,$$
\[
- \dot{\eta}^* - g(\eta + H)q_{i,i}^* - q_i^* H_i \frac{1}{\bar{\xi}^2} q_i q_j q_j^* + Q(\eta - \eta^{ob}) = 0 \quad \text{in} \quad \Omega, \tag{32}
\]
\[
q_i^*(t_f) = 0 \quad \text{in} \quad \Omega, \tag{33}
\]
\[
\eta^*(t_f) = 0 \quad \text{in} \quad \Omega, \tag{34}
\]
\[
\left\{ (\eta + H)q_i^* - \frac{1}{\bar{\xi}^2} q_i q_j q_j^* \right\} n_i = 0 \quad \text{on} \quad \Gamma_D, \tag{35}
\]
\[
\{ \text{in} \Omega \}. \tag{36}
\]

The gradient of the extended performance function with respect to the control variable can be obtained, and it is expressed by the following form.

\[
Grad(J^*)_i = \left\{ \eta + \frac{1}{\bar{\xi}} q_j q_j^* \right\} n_i \quad \text{on} \quad \Gamma_C. \tag{37}
\]

5 Minimization Technique

5.1 Weighted Gradient Method

In this study, the weighted gradient method is applied as the minimization technique. In the weighted gradient method, a modified performance function \( K \) to which a penalty term is added is used and expressed by the following equation:

\[
K^{(l)} = J^{(l)} + \frac{1}{2} \int_{t_0}^{t_f} \int_{\Gamma_C} (U_{i}^{(l+1)} - U_{i}^{(l)}) W_{ij}^{(l)} (U_{j}^{(l+1)} - U_{j}^{(l)}) d\Gamma_C dt, \tag{38}
\]

where \( l \) and \( W_{ij}^{(l)} \) are number of iteration and the weighting parameter, respectively. The penalty term will be zero, in case that the modified performance function is converged to the minimum value. The minimization of the performance function is equal to minimize the modified performance function. The following stationary condition is used to obtain the minimum value of the modified performance function

\[
\delta K^{(l)} = 0. \tag{39}
\]

The control discharge is updated by the following equation.

\[
W^{(l)} U_{i}^{(l+1)} = W^{(l)} U_{i}^{(l)} - Grad(J^*)_i^{(l)} \tag{40}
\]

5.2 Algorithm

The following algorithm is employed for the computation.

Step 1. Chose the initial control discharge \( U^{(0)} \).

Step 2. Solve the state variables \( q_{i}^{(0)} \) and \( \eta^{(0)} \) using eqs.(1)-(2).

Step 3. Compute the initial performance function \( J^{(0)} \).

Step 4. Solve \( q_{i}^{(l)} \) and \( \eta^{(l)} \) using eqs.(31)-(32).

Step 5. Solve the gradient using eq.(37).

Step 6. Update the control discharge \( U_{i}^{(l)} \) using eq.(40).

Step 7. Solve the state variables \( q_{i}^{(l+1)} \) and \( \eta^{(l+1)} \) with the control discharge using eqs.(1)-(2).
Step 8. Compute the performance function $J^{(l+1)}$.

Step 9. Solve $q_{i}^{*(l+1)}$ and $y_{i}^{*(l+1)}$ using eqs.(31)-(32).

Step 10. Solve the gradient using eq.(37).

Step 11. Update the control discharge $U_{i}^{(l+1)}$ using eq.(40).

Step 12. Check the convergence; if $||U_{i}^{(l+1)} - U_{i}(l)|| < \epsilon$ then stop, else go to step 13.

Step 13. Update a weighting parameter $W_{ij}^{(l)}$;
   if $J^{(l+1)} < J^{(l)}$, then set $W_{ij}^{(l+1)} = 0.9W_{ij}^{(l)}$ and go to step 4,
   else $W_{ij}^{(l+1)} = 2.0W_{ij}^{(l)}$ and go to step 7.

6 Verification

In order to confirm that the introduced technique is correct, the verification is carried out using a simple rectangular channel model. The finite element mesh which has 303 nodes and 400 elements is shown in Figure 4. Total length and width are 100[m] and 2[m], respectively. The water depth is 10[m] at whole domain. The time increment $\Delta t$ and the total time are set as 0.02[s] and 10[s]. The inflow boundary, the control boundary and the target points are set on the left, on the right and on the center of computational domain, respectively. On the inflow boundary, the sin wave water discharge is given as Figure 5. The object value of the water elevation at the target points is set to 0.0[m]. The problem is to find the optimal control discharge on the control boundary so as to close the water elevation at the target points to object value.

Figure 6 shows variation of the performance function. The performance function converges to 0.0. Figure 8 shows the time history of the water elevation at the target point without control and with control. It is seen that the water elevation at target points is reduced to object value. Figure 7 shows the time history of the control discharge.

Therefor, the optimal control of the water elevation at the target points in the simple model is carried out.

Figure 4: Finite element mesh, 303 nodes, 400 elements
7 Numerical Study

In this study, optimal control of water elevation in the Ikari dam lake which is located in Tochigi prefecture in Japan is carried out. The Ikari dam lake shown in Figure 10 impounds the water that has flowed from the Oga river in the upstream, and discharge to the Kinu river in the downstream. There is the hot springs resort area in the downstream as shown in Figure 9. In case that a flood occurs in upstream of the Ikari dam and the much water is discharged to the downstream, the hot springs resort area suffers damage. Then, there is the Kawaji dam close to the Ikari dam, and these two dam lakes can exchange water through the aqueduct. The Ikari dam lake has the pump system at the aqueduct which can send the water that does not use in the Ikari dam lake and the water exceed the capacity of the dam. If the over water volume will be discharged from the Ikari dam, it has the possibility of damaging the hot springs resort area. To reduce the risk, it is necessary to determine the volume to send the water to the Kawaji dam lake using the aqueduct. In this study, the optimal water discharge sent by pump system of the aqueduct to minimize the water elevation in the Oga river is carried out.

The finite element mesh is shown in Figure 11. The average of element size is about 10[m]. The total number of nodes and elements is 2876 and 5215, respectively. The water depth is shown in Figure 12. The time increment is set as 0.1[s]. Total duration time step is 1500. Total time is 1[h]. The inflow boundary, the outflow boundary, the control boundary and the target point are set as shown in Figure 2. On the inflow boundary, the boundary conditions are assumed as flood occurred in September, 2011 as shown in Figure 13. The object value of water elevation is set to 0.0[m]. The initial condition of the discharge and the water elevation are set 0.0[m/s] and 0.0[m]. The control discharge on the control boundary is computed by the method presented in this study.
8 Numerical Results

Figure 14 shows variation of the performance function. The performance function converges to about 0.6. Figure 15 shows the time history of the control discharge. Figure 16 shows the water elevation at the target point without control and with control. It is seen that the water elevation at target point become lower than without control.

By these results, the optimal discharge at the control point to minimize the water elevation at the target point is performed.

9 Conclusion

In this research, the optimal control of the water elevation in the shallow water flow using the first-order adjoint equation method is presented. The control discharge is derived so as to minimize the performance function using the optimal control theory presented in this paper. The water elevation can be as close as object value. As the future works, determination of the optimal time duration will be presented.
References


Figure 10: The Ikari dam lake

Figure 11: Finite Element Mesh

Figure 12: Water depth
Figure 13: Inflow water elevation

Figure 14: Performance function

Figure 15: Control discharge

Figure 16: Water elevation at target point
Figure 17: Water elevation without control

Figure 18: Water elevation with control