Optimal Control of Water Level Using Time-Delay system

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Abstract

This paper presents an optimal control problem using time-delay system based on the optimal control theory. The time-delay system includes two types of control, ordinary and delayed controls. On the problem of drainage basin, ordinary control denotes the control on which the water drains out into the drainage basin, and delayed control denotes the control on which the water returns to the river. Each control values are expressed as the same function form. In this paper, optimal control of water level is considered using a model of Tsurumi river. As the spatial discretization, the Galerkin method is applied. The bubble function element is used to the interpolation function. The Crank-Nicolson method is applied to the temporal discretization. As the minimization technique of the performance function, the Sakawa-Shindo method is utilized.

Keywords : Time-Delay System, Drainage Basin, Optimal Control Theory, Bubble Function, Sakawa-Shindo Method

1 Introduction

A lot of natural disasters have happened in Japan. Disasters concerned with natural water are serious problems in Japan where heavy rain falls. Flood is one of the serious problem because many Japanese rivers have narrow width and steep inclination. Moreover, the urbanization in recent years causes increment of water amount that flows to river. Drainage basin is one of the prevention for flood. The drainage basin is big area constructed next to the river, and it changes itself as extraordinary reserve area when flood occurs. If the water level of the river rises by flood, the water is drained out into the drainage basin, and the water is returned to the river bit by bit when the water level of the river sinks down.

On the problem of drainage basin, it is necessary to include the drainage basin area and river area as the computational domain. However, much calculation time and computational memory is needed to analyze. In this paper, the time-delay system is employed. The time-delay system is useful theory for treating the problem of drainage basin because past operation can be considered by regarding the control value and the delay control value as the same function form using the parameters $\tau$ and $\alpha$. The normal control value $U_i(t)$ denotes the velocity of water which drains out into the drainage basin. The delay control value $U_i(t-\tau)$ denotes the velocity of water which returns to the river after $\tau$ passed. The formulation of the time-delay system based on the optimal control theory can be obtained by adding the time-delay term $U_i(t-\tau)$. It is simple to introduce the time-delay system into the optimal control theory. Moreover, on the time-delay system, it is not necessary to consider the drainage basin area as the computational domain, so it is possible to analyze at less calculation time and with less computational memory. For these reasons, it can be said that the time-delay system is effective theory on the problem of drainage basin. In this paper, the main purpose is set to control the water level of river using drainage basin, which can be modeled by the time-delay system.

As the governing equation, the shallow water equation is applied. For the spatial discretization, the Galerkin method using bubble function element is used. The stabilized bubble function is employed. Optimal control theory using the Lagrange multiplier method is applied for minimizing the performance function. For the minimization technique, the Sakawa-Shindo method is utilized. As a numerical example, the Tsurumi river which locates in Kanagawa Prefecture is used.
2 State Equation

2.1 Shallow Water Equation

The non-linear shallow water equation is applied to calculate the water behavior. The shallow water equation consists of the momentum equation and the continuity equation, which can be written as follows:

\[
\begin{align*}
\dot{u}_i + u_j u_{i,j} + g(\xi + \eta + h)_i - \nu(u_{i,j} + u_{j,i})_j + fu_i &= 0, \\
\dot{\xi} + ((\xi + h)u_i)_i &= 0,
\end{align*}
\]

where \(u_i, g, \xi, \eta\) and \(h\) are the water velocity, the gravitational acceleration, the water elevation, the bed elevation and the water depth respectively. The coefficient of kinematic eddy viscosity \(\nu\) and the bottom friction \(f\) is expressed as follows:

\[
\begin{align*}
\nu &= \frac{k_l}{6} u_s(h + \xi), \\
f &= \frac{u_s}{h + \xi},
\end{align*}
\]

where \(k_l\) and \(u_s\) are the Karman constant and the friction velocity respectively. The friction velocity is expressed as follows:

\[
u_s = \frac{gn^2 \sqrt{u_k u_k}}{(h + \xi)^{3/2}},\]

where \(n\) is the Manning roughness coefficient. The initial conditions are given as follows:

\[
\begin{align*}
u_i(t_0) &= \hat{u}_{i0} \quad \text{in} \quad \Omega, \\
\xi(t_0) &= \hat{\xi}_0 \quad \text{in} \quad \Omega.
\end{align*}
\]

The boundary conditions are given as follows:

\[
\begin{align*}
\nu_i &= \hat{u}_i \quad \text{on} \quad S_1, \\
\xi &= \hat{\xi} \quad \text{on} \quad S_1, \\
u_i n_i &= \hat{u}_n \quad \text{on} \quad S_2.
\end{align*}
\]

2.2 Control Boundary

The following values are given on the control boundaries:

\[
\begin{align*}

u_i = U_i(t) & \quad \text{on} \quad S_{cont}, \\

u_i = -\alpha U_i(t - \tau) & \quad \text{on} \quad S_{del},
\end{align*}
\]

where \(\tau\) and \(\alpha\) are delayed time and amplitude parameter, respectively. \(S_{cont}\) is the ordinary control boundary at which the water is drained out into the drainage basin, \(S_{del}\) is the delayed control boundary at which the water is discharged to the river.

3 Spatial Discretization

3.1 Bubble Function Element

As for the spatial discretization, the Galerkin method is applied. The bubble function element is applied as the interpolation function. The bubble function interpolation is expressed as follows:

\[
\begin{align*}
\nu_i &= \Phi_1 u_i + \Phi_2 u_{i2} + \Phi_3 u_{i3} + \Phi_4 u_{i4}, \\
\hat{u}_{i4} &= u_{i4} - \frac{1}{4} (u_{i1} + u_{i2} + u_{i3}), \\
\xi &= \Phi_1 \xi_1 + \Phi_2 \xi_2 + \Phi_3 \xi_3 + \Phi_4 \xi_4, \\
\hat{\xi}_4 &= \xi_4 - \frac{1}{3} (\xi_1 + \xi_2 + \xi_3), \\
\Phi_1 &= L_1, \quad \Phi_2 = L_2, \quad \Phi_3 = L_3, \quad \Phi_4 = 27 L_1 L_2 L_3,
\end{align*}
\]

where \(\Phi_\alpha (\alpha = 1, 2, 3, 4)\) is the bubble function in four-nodes triangular elements. The continuous bubble function is used and is shown in Fig.1
3.2 Stabilized Bubble Function

The bubble function is capable of eliminating the barycenter point by regarding the static condensation. The discretized form derived from the bubble function element is equivalent to form by SUPG method. Therefore, the stabilized parameter for the momentum equation and the continuity equation derived from the bubble function element is expressed as follows;

\[ \tau_e B = \frac{1}{2} \left[ \phi_e \right] \frac{1}{A_e} \left( \phi_e, 1 \right)_\Omega A_e^{-1} \left( \frac{1}{h_e}, \frac{1}{h_e} \right) \frac{1}{2} \left( \left( \nu + h \right) \| \phi_e \|_{H^1}^2 + f \| \phi_e \|_\Omega^2 \right) \frac{1}{2} \left( \left( \nu + h \right) \| \phi_e \|_{H^1}^2 + f \| \phi_e \|_\Omega^2 \right) \]

where \( \tau_e \) is the stabilized control parameter. From the stabilized parameter comparable to the SUPG method, an optimal parameter can be given as follows;

for the momentum equation,

\[ \tau_e B_{u_i} = \frac{1}{2} \left( \frac{\phi_e}{h_e} \right)^2 \left( \left( \frac{U_i}{h_e} \right)^2 + \left( \frac{4 \nu}{h_e} \right)^2 + \left( \frac{u_s}{\left( \xi + h \right)} \right)^2 \right) \frac{1}{2} \]

and for the continuity equation,

\[ \tau_e B_{\eta} = \frac{1}{2} \left( \frac{\phi_e}{h_e} \right)^2 \left( \left( \frac{U_i}{h_e} \right)^2 + \left( \frac{4 \nu}{h_e} \right)^2 + \left( \frac{u_s}{\left( \xi + h \right)} \right)^2 \right) \frac{1}{2} \]

where \( \alpha = \frac{A_e \| \phi_e \|^2_{H^1}}{\left( \phi_e, 1 \right)_\Omega}, \) \( h_e = \sqrt{2A_e}, \) \( |U_i| = \sqrt{u^2 + v^2 + g(\xi + h)}. \)

3.3 Temporal Discretization

As for the temporal discretization, the implicit scheme which is capable of taking long time increment and excels in stability is used. The implicit scheme is suitable for treating large scale problem. The Crank-Nicolson method is applied to the state equation and adjoint equation.

4 Optimal Control

4.1 Performance Function

The optimal control of time-delay system is mentioned in this section. The performance function is defined as follows;

\[ J = \frac{1}{2} \int_{t_0}^{t_f} \int_{\Omega} \left( \xi - \xi_{obj} \right)^T Q \left( \xi - \xi_{obj} \right) d\Omega dt, \]

where \( Q \) is weighting diagonal matrix, \( \xi \) and \( \xi_{obj} \) are the state value and the target value at object points, respectively. The superscript \( T \) denotes transpose.
4.2 Adjoint Equation

The main purpose is to find the optimal control value so as to minimize the performance function. The Lagrange multiplier method is applied to the minimization problem of the performance function. By the Lagrange multiplier and the state equation, the performance function is extended. The extended performance function can be written as follows;

\[ J^* = J + \int_{t_0}^{t_f} \int_{\Omega} \lambda_u^T [\dot{u}_i + u_j S_j u_i + g(\xi + \eta + h)_i + \nu \{ H_{ji} u_i + H_{ji} u_j \} + f M u_i] d\Omega dt \]

\[ + \int_{t_0}^{t_f} \int_{\Omega} \lambda_{\xi}^T [\dot{\xi} + u_i S_i (\xi + h) + (\xi + h) S_i u_i] d\Omega dt \]

\[ + \int_{t_0}^{t_f} \int_{S_{del}} \lambda_u^T [-\alpha u_j S_j U_i (t - \tau) + \nu \{ -\alpha H_{ji} U_i (t - \tau) - \alpha H_{ji} U_j (t - \tau) \} + f \{ -\alpha M U_i (t - \tau) \}] dS_{del} dt \]

\[ + \int_{t_0}^{t_f} \int_{S_{del}} \lambda_{\xi}^T [-\alpha (\xi + h) S_i U_i (t - \tau)] dS_{del} dt \], (23)

where \( \lambda_u \) and \( \lambda_{\xi} \) denote the Lagrange multiplier for the velocity and the water elevation, respectively. The extended performance function is expressed by the Hamiltonian and time differential term.

\[ J^* = \int_{t_0}^{t_f} \int_{\Omega} (H + \lambda_u^T \dot{u}_i + \lambda_{\xi}^T \dot{\xi}) d\Omega dt, \] (24)

where the Hamiltonian \( H \) is expressed as follows;

\[ H = \frac{1}{2} (\xi - \xi_{obj})^T Q (\xi - \xi_{obj}) \]

\[ + \lambda_u^T [\dot{u}_i + u_j S_j u_i + g(\xi + \eta + h)_i + \nu \{ H_{ji} u_i + H_{ji} u_j \} + f M u_i] \]

\[ + \lambda_{\xi}^T [\dot{\xi} + u_i S_i (\xi + h) + (\xi + h) S_i u_i] \]

\[ + \lambda_u^T [-\alpha u_j S_j U_i (t - \tau) + \nu \{ -\alpha H_{ji} U_i (t - \tau) - \alpha H_{ji} U_j (t - \tau) \} + f \{ -\alpha M U_i (t - \tau) \}] \]

\[ + \lambda_{\xi}^T [-\alpha (\xi + h) S_i U_i (t - \tau)]. \] (25)

To obtain the control value which minimizes the performance function, the stationary condition \( \delta J^* = 0 \) is needed. Under this condition and boundary condition (11), the adjoint equation and terminal condition which is the same as without using time-delay system can be obtained as follows;

\[ \frac{\partial H}{\partial u_i} - \lambda_u^T M = 0, \quad \frac{\partial H}{\partial \xi} - \lambda_{\xi}^T M = 0, \]

\[ \lambda_u^T (t_f) = 0, \quad \lambda_{\xi}^T (t_f) = 0. \] (26)

5 Minimization Technique

5.1 Sakawa-Shindo Method

The Sakawa-Shindo method is applied to minimize the performance function. By adding a penalty term to the extended performance function, the extended performance function is modified, which is expressed as follows;

\[ K = J^{*(l)} + \frac{1}{2} \int_{t_0}^{t_f} (U_i^{(l+1)} - U_i^{(l)})^T W^{(l)} (U_i^{(l+1)} - U_i^{(l)}) dt, \] (28)

where \( l \) is the iteration number, \( W^{(l)} \) is the weighting diagonal matrix and \( U_i \) is the control velocity. Considering the stationary condition \( \delta K = 0 \), the renewal equation of control velocity is obtained as follows;
\[
U_i^{(l+1)} = U_i^{(l)} - \frac{1}{W(l)} \left( \frac{\partial H}{\partial U_i} + \frac{\partial H}{\partial U_i} \bigg|_{t=t+\tau} \right) \quad t_0 \leq t \leq t_f - \tau, \\
U_i^{(l+1)} = U_i^{(l)} - \frac{1}{W(l)} \left( \frac{\partial H}{\partial U_i} \right) \quad t_f - \tau < t \leq t_f, 
\]

(29) (30)

where \( \frac{\partial H}{\partial U_i} \) and \( \frac{\partial H}{\partial U_i} \bigg|_{t=t+\tau} \) are denoted the gradient with respect to the control velocity and expressed as follows;

\[
\frac{\partial H}{\partial U_i} = \frac{\partial}{\partial U_i} H(t, u_i(t), \xi(t), \lambda_u(t), \lambda_\xi(t), U_i(t), U_i(t - \tau)), \\
\frac{\partial H}{\partial U_i} \bigg|_{t=t+\tau} = \frac{\partial}{\partial U_i} H_\tau(t + \tau, u_i(t + \tau), \xi(t + \tau), \lambda_u(t + \tau), \lambda_\xi(t + \tau), U_i(t + \tau), U_i(t)).
\]

(31) (32)

### 5.2 Algorithm

The algorithm of the Sakawa-Shindo method is shown as follows;

1. Set initial control values \( U_i^{(l)} \) and set \( l = 0 \).
2. Obtain \( u_i^{(l)} \) and \( \xi^{(l)} \) by solving the state equations (1) and (2).
3. Obtain \( \lambda_u^{(l)} \) and \( \lambda_\xi^{(l)} \) by solving the adjoint equation (26).
4. Compute modified control values \( U_i^{(l+1)} \).
5. Obtain \( u_i^{(l+1)} \) and \( \xi^{(l+1)} \) by solving the state equations (1) and (2).
6. If \( J^{(l)} - J^{(l+1)} < \varepsilon \), then stop, else go to 7.
7. If \( J^{(l+1)} < J^{(l)} \), then set \( W^{(l+1)} = 0.9W^{(l)} \), \( l = l + 1 \), and go to 3, else \( W^{(l+1)} = 2.0W^{(l)} \) and go to 4.

### 6 Numerical Study

As a numerical study, optimal control of water elevation is carried out. In this paper, Tsurumi river which is located Kanagawa Prefecture is used as shown in Fig.2. This river has a sudden curve, therefore, the water overflows easily at the curving point. Optimal control using drainage basin modeled by the time-delay system is carried out to reduce the water elevation at the curving point. The length of analytical area is about 2(km). The finite element mesh is shown in Fig.3. Total number of nodes and elements are 1106 and 1782, respectively. On the edge of upstream side of the river, inflow velocity which is observed in September 1996 is given as shown in Fig.4. The delayed time is set to 10(hours) and the amplitude parameter is set to 0.1. The Manning roughness coefficient, the Karman constant, and calculation time are 0.03(sec/m\(^1/3\)), 0.41, and 24(hours), respectively.

Fig.5 shows the control velocity on the control boundary and delay boundary. Fig.6 shows the water elevation at object point. As shown in Fig.6, the maximum value of water level is reduced.

The amplitude parameter is changed from 0.1 to 0.5. Table 1 shows the comparison of maximum water level at object point and maximum control velocity on the control boundary in each amplitude parameters.

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Figure 2: Tsurumi River

Figure 3: Finite Element Mesh
7 Conclusion

In this paper, the extended time-delay system based on the optimal control theory is presented. The bubble function interpolation can be applied. The optimal control theory can be extended to the time-delay system. The Sakawa-Shindo method can be applied to the time-delay system. As the numerical example, using a model of Tsurumi river, the optimal control velocity and the delayed control velocity can be obtained. The maximum water level can be reduced. The relationship between the amplitude parameter and the peak value is shown.

References

