Kalman Filter Finite Element Method

1 INTRODUCTION

This chapter presents the Kalman filter finite element method. The numerical simulation is frequently applied as the computer develops and analytical technology improves. The numerical simulation is paid attention as an effective scheme for the estimation of natural phenomenon. One of method is the finite element method (FEM)\(^9\)[10]. However, it is very difficult from the following reasons to apply numerical analysis to actual problems.

- The basic equation used for numerical analysis cannot express an actual phenomenon completely, and includes various errors.
- The observation data obtained by actual observation includes mechanical and individual errors.
- It is economically and physically very difficult to establish many observation points.
- It is difficult to dispose boundary condition.

In order to solve these problems, the Kalman filter finite element method is employed. This method is combination of the finite element method and the Kalman Filter (KF). KF \(^1\)[2][3][4][5] is the filtering algorithm presented by Kalman and Bucy in 1960’s which is based on the stochastic process theory of the state space model and the orthogonal projection for the linear system. In the KF, the system and observation noise are included. The system noise is error that arises in approximating basic equation. On the other hand, the observation noise includes artificial and mechanical errors at the observation data. The KF is able to estimate the state value by eliminating these noises. However, KF cannot estimate the state value in space direction. Thus, by applying the finite element equation to the state transition matrix in state-space model, KF and FEM can be combined. The KF-FEM is capable of estimating the state value considering not only in time but also in space directions.

In this Kawahara laboratory, the estimation of the density of pollutant, estimation of ground temperature control system, and so on \(^6\)[7][8] has been researched by using KF-FEM. In numerical example, the tidal current is estimated by using shallow water equation as a state equation. The effectiveness of KF-FEM is verified by weighing the estimation value against the observation value.
2 THE KALMAN FILTER

2.1 State Space Model

The Kalman Filter is based on a set of two systems. The system equation can be expressed state of the phenomena. The observation equation is denoted relation between the actual observation data and the state value.

The system equation is

\[ x_{k+1} = F_k x_k + G_k w_k, \]  

(1)

and the observation equation is

\[ y_k = H_k x_k + v_k, \]  

(2)

where \( x_k \) is state vector at time \( k \), \( F_k \) is state transition matrix which represents the basic relation of the phenomenon, \( G_k \) is driving matrix and \( w_k \) is a system noise, \( y_k \) is observation vector at time \( k \), \( H_k \) is observation matrix and \( v_k \) is an observation noise, respectively.

The system noise \( w_k \) is assumed as:

\[ E\{w_k\} = 0, \]  

(3)

\[ \text{cov}\{w_k, w_j\} = E\{w_k, w_j^T\} = Q_k \delta_{kj}, \]  

(4)

and the observation noise \( v_k \) is also assumed as:

\[ E\{v_k\} = 0, \]  

(5)

\[ \text{cov}\{v_k, v_j\} = E\{v_k, v_j^T\} = R_k \delta_{kj}, \]  

(6)

with

\[ E\{w_k, v_j\} = 0, \]  

(7)

where \( \delta_{kj} \) is the Kronecker’s delta function.

\[ \delta_{kj} = \begin{cases} 1 & k = j \\ 0 & \text{otherwise} \end{cases} \]

in which \( E \) is an average operator, \( Q \) is system error covariance, and \( R \) is observation error covariance, respectively.
2.2 Assumption

The optimal estimate $\hat{x}_k$ is the average of $x_k$ giving the observation data $Y_k$,

$$\hat{x}_k = E\{x_k \mid Y_k\}. \quad (8)$$

The covariance $P_k$ is written as follows:

$$P_k = cov\{x_k \mid Y_k\} = E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\}. \quad (9)$$

$P_k$ is called estimated error covariance.

The estimate $x_k^*$ is the average of $x_k$ giving the observation data $Y_{k-1}$,

$$x_k^* = E\{x_k \mid Y_{k-1}\}. \quad (10)$$

The covariance $\Gamma_k$ is written as follows:

$$\Gamma_k = cov\{x_k \mid Y_{k-1}\} = E\{(x_k - x_k^*)(x_k - x_k^*)^T\}. \quad (11)$$

$\Gamma_k$ is called predicted error covariance.

The whole process is assumed as the stochastic phenomenon.

2.3 Formulation

The Bayes rule is shown as follows:

$$P(x_k \mid Y_k) = \frac{P(y_k \mid x_k)P(x_k \mid Y_{k-1})}{P(y_k \mid Y_{k-1})}. \quad (12)$$

Optimal estimated value $\hat{x}_k$, Kalman-gain $K_k$, estimated error covariance $P_k$ and predicted error covariance $\Gamma_{k+1}$ are derived from the above assumptions;

$$\{\hat{x}_k\} = \{x_k^*\} + [K_k]\{(y_k - [H]\{x_k^*\})\},$$

$$[K_k] = \Gamma_k[H]^T([R] + [H]\Gamma_k[H]^T)^{-1},$$

$$[P_k] = ([I] - [K_k][H])\Gamma_k,$$

$$[\Gamma_{k+1}] = [F][P_k][F]^T + [G][Q][G]^T,$$

where $Q$ is system error covariance and $R$ is observation error covariance, respectively.
3 FINITE ELEMENT METHOD

3.1 Basic Equation

In this research, basic equation is employed for the linear shallow water equation as follows;

\begin{align}
\dot{u}_i + g \eta, i &= 0, \\
\dot{\eta} + hu_{i,i} &= 0,
\end{align}

where \( u_i \) is water velocity of \( X \) and \( Y \) direction, \( \eta \) is water elevation, \( h \) is water depth, and \( g \) is the gravitational acceleration, respectively.

3.2 Boundary Condition

In the boundary \( \Gamma \), boundary conditions are given as follows;

\[ u_n = u_i n_i = \hat{u} \text{ on } \Gamma_n, \]

where \( \hat{u} \) means the specified value on the boundary and \( n_i \) denotes the direction cosines of the unit outward normal of the boundary, respectively.

3.3 Discretization

The discretization in space is carried out applying the Galerkin finite element method. The velocity and water elevation on a triangle element interpolated using linear interpolation function. For the weighting function the same function as the interpolation function is employed. The explicit Euler method is applied to discretization in time.

The finite element equation is shown as follows;

\begin{align}
\tilde{M}_{\alpha\beta} u_{\beta}^{n+1} &= \tilde{M}_{\alpha\beta} u_{\beta}^{n} - \Delta t g S_{\alpha\beta,x} \eta_{\beta}^{n}, \\
\tilde{M}_{\alpha\beta} v_{\beta}^{n+1} &= \tilde{M}_{\alpha\beta} v_{\beta}^{n} - \Delta t g S_{\alpha\beta,y} \eta_{\beta}^{n}, \\
\tilde{M}_{\alpha\beta} \eta_{\beta}^{n+1} &= \tilde{M}_{\alpha\beta} \eta_{\beta}^{n} - \Delta t h (S_{\alpha\beta,x} u_{\beta}^{n} + S_{\alpha\beta,y} v_{\beta}^{n}),
\end{align}

where \( \tilde{M} \) is the selective lumping coefficient and written as;

\[ \tilde{M} = e \tilde{M} + (1 - e) M, \]

where \( e \) is the lumping parameter adjusting the stability of computation.
4 THE KALMAN FILTER FINITE ELEMENT METHOD

4.1 State Transition Matrix $F_k$

A physical model is not considered in the conventional Kalman Filter, and spatial distribution of the state cannot be estimated only by the time series. Then, the estimation of spatial distribution of the state is enabled by taking the finite element method into Kalman Filter. The finite element equations is applied to the state transition matrix in Kalman Filter. From the finite element equations eqs.(15) – (18), the state transition matrix is given as:

$$
\begin{bmatrix}
\hat{u}_\beta \\
\hat{v}_\beta \\
\hat{\eta}_\beta 
\end{bmatrix}^{n+1} = \bar{M}_{\alpha\beta}^{-1} \begin{bmatrix}
\hat{M}_{\alpha\beta} & -\Delta t g S_{\alpha\beta,x} \\
\hat{M}_{\alpha\beta} & -\Delta t g S_{\alpha\beta,y} \\
-\Delta t h S_{\alpha\beta,x} & -\Delta t h S_{\alpha\beta,y} 
\end{bmatrix} \begin{bmatrix}
\hat{u}_\beta \\
\hat{v}_\beta \\
\hat{\eta}_\beta 
\end{bmatrix}^n,
$$

where the coefficient matrix is represented by $F_k$ and can be used in eq.(8).

This matrix is a stationary state independent of time series.

4.2 Algorithm

The algorithm of the Kalman Filter Finite Element Method is written as follows.

1. $[\Gamma_0] = [v_0]$ \{\hat{x}_{-1}\} = \{\hat{x}_0\}.
2. $[K_k] = [\Gamma_k][H]^T([R] + [H][\Gamma_k][H]^T)^{-1}.$
3. $[P_k] = ([I] - [K_k][H])[\Gamma_k].$
4. $[\Gamma_{k+1}] = [F][P_k][F]^T + [G][Q][G]^T.$
5. if $|tr[P_k] - tr[P_{k-1}]| < \varepsilon,$
   \begin{align*}
   \text{then} & \quad \text{go to 6,} \\
   \text{else} & \quad \text{go to 2.}
   \end{align*}
6. $\{x^*_n\} = [F]\{\hat{x}_{n-1}\}.$
7. $\{\hat{x}_n\} = \{x^*_n\} + [K_n](\{y_n\} - \{H\}{x^*_n}).$

Although originally calculated by one framework, it can divide into two contents in this way. Because linear equation is used as a basic equation of a physical model in this research. From 2 to 5 is called off-line and is calculation independent of an observation value. On the other hand, 6 and 7 are called on-line and calculation depending on observation value. In this way, the calculation time can be shortened. Off-line time step is represented by $k$ and on-line time step is by $n$. In this chapter, the state vector $x$ expresses the velocity $u$, $v$ and water elevation $\eta$. 
5 NUMERICAL EXAMPLE

5.1 COMPUTATIONAL MODEL

Measurement at the Onjuku Coast in Japan is used as a numerical simulation. Fig.1 shows the finite element mesh and placement of observation points. Total number of nodes and elements are 600 and 1097. Water depth is expressed in Fig.2.

![Finite Element Mesh](image1.png)

**Fig.1; Finite Element Mesh**

![Water Depth](image2.png)

**Fig.2; Water Depth**
5.2 OBSERVATION DATA

Flow velocity in x and y direction and water elevation were actually observed respectively at five points in the Onjuku Coast. Fig.3 shows arrangement of each observation point. The data of five days from July 16, 1997 to the 20th was used and analyzed as the observational data. Figs.4-19 are that the data used for the analysis of velocity in x and y direction and water elevation at each observation point.

Fig.3; Observation Point
Fig. 4: x-Velocity at No. 1

Fig. 5: y-Velocity at No. 1

Fig. 6: Water Elevation at No. 1
Fig. 7; x-Velocity at No. 2

Fig. 8; y-Velocity at No. 2

Fig. 9; Water Elevation at No. 2
Fig. 10: \( u \) - Velocity at No. 3

Fig. 11: \( v \) - Velocity at No. 3

Fig. 12: \( \eta \) - Water Elevation at No. 3
Fig. 13; x-Velocity at No. 4

Fig. 14; y-Velocity at No. 4

Fig. 15; Water Elevation at No. 4
Fig. 16; x-Velocity at No.5

Fig. 17; y-Velocity at No.5

Fig. 18; Water Elevation at No.5
5.3 Estimation of Tidal Current

Tidal Current (velocity in x direction, velocity in y direction, and water elevation) is estimated using Kalman Filter Finite Element Method. Time increment $\Delta t$ is 1.0(s), lumping parameter $e$ is 0.9, observation error covariance $R$ is $1.0 \times 10^{-3}$ and system error covariance $Q$ is $1.0 \times 10^{-3}$, respectively.

5.4 NUMERICAL RESULT

Figs.19-33 show the velocity distribution in the entire analytical domain. In this current estimation, the observation value in four points (No.1 ~ No.4) were used as input data. And, estimation value and the observation value were compared by observation data at point No.5, which is not used in the analysis, and the validity of Kalman Filter Finite Element Method was verified. Figs.35 ~ 37 show the comparison between estimation value and the observation value of velocity in x direction, velocity in y direction, and water elevation at No.5.
Fig. 19; Velocity Distribution (0:00AM, July 16th)

Fig. 20; Velocity Distribution (8:00AM, July 16th)

Fig. 21; Velocity Distribution (4:00PM, July 16th)
Fig.22; Velocity Distribution (0:00AM, July 17th)

Fig.23; Velocity Distribution (8:00AM, July 17th)

Fig.24; Velocity Distribution (4:00PM, July 17th)
Fig. 25; Velocity Distribution (0:00AM, July 18th)

Fig. 26; Velocity Distribution (8:00AM, July 18th)

Fig. 27; Velocity Distribution (4:00PM, July 18th)
Fig. 28; Velocity Distribution (0:00AM, July 19th)

Fig. 29; Velocity Distribution (8:00AM, July 19th)

Fig. 30; Velocity Distribution (4:00PM, July 19th)
Fig. 31; Velocity Distribution (0:00AM, July 20th)

Fig. 32; Velocity Distribution (8:00AM, July 20th)

Fig. 33; Velocity Distribution (4:00PM, July 20th)
Fig. 34; Comparison between estimation value and observation value (x-velocity) at No. 5

Fig. 35; Comparison between estimation value and observation value (y-velocity) at No. 5

Fig. 36; Comparison between estimation value and observation value (water elevation) at No. 5
6 CONCLUSION

In this chapter, the Kalman Filter Finite Element Method is presented. Using this method, the computational simulation applying the state-space problem including the effect of observation and physical model has been carried out. The purpose of this research is to propose the Kalman Filter Finite Element Method as a estimation technique, and to verify the validity of this method. In numerical example, tidal current at the Onjuku Coast is estimated by this method. The distribution of the entire domain is able to be estimated from the limited data (from 4 observation points). As accuracy of estimation value, water elevation shows good result. The Kalman Filter Finite Element Method has been shown as the useful method for the analysis considering observation.

References