A Fictitious Domain Method
for Incompressible Viscous Flow Around Moving Particles

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ABSTRACT

This paper discusses the numerical solution of the incompressible viscous flow by the fictitious domain method with distributed Lagrange multiplier. For the analysis of the finite element method, the mixed interpolation based on the bubble function is presented. How to use the fictitious domain method applied to the Navier-Stokes equation is explained. The incompressible viscous flow is restricted by the distributed Lagrange multiplier method. Advantage of the fictitious domain method based on distributed Lagrange multiplier is investigated.

KEY WORDS

finite element method, fictitious domain method, distributed Lagrange multiplier, Navier-Stokes equations, Newton second law

1 INTRODUCTION

In recent years, because of rapid development of the finite element method and computer hardware, complicated and large-scale computation can be performed. The finite element method is useful tool for the moving boundary analysis. But generally remeshing is required at each time cycle in case of the moving boundary problem by the conventional finite element method. Remeshing procedure has a lot of problems on the complicated and large-scale computation, such as a shortage of memory and increase of calculation time. Thus, the important feature is to avoid the remeshing. In the fictitious domain method 1\textsuperscript{-}8, two kinds of finite element meshes are used. One is objective domain that is restricted by the Lagrange multiplier, the other mesh is for whole computational domain. It is possible to move the objective domain in the whole computational domain without fitting the nodes of boundary. For the reason, reduction of the memory and shortening of computational time are drastically possible. Another advantage of the fictitious domain method is not necessary to compute fluid forces, because in the fictitious domain method defines the Lagrange multiplier as fluid force. In this research, the efficiency of moving boundary problems by the fictitious domain method is investigated.
2 NAVIER STOKES EQUATIONS

As the basic equation of the incompressible viscous fluid, the incompressible Navier-Stokes equations is used which is expressed as;

\[
\frac{\partial u_i}{\partial t} + u_j u_{i,j} + p = f_i \quad \text{in} \quad \Omega \setminus \omega, \\
u_{i,i} = 0 \quad \text{in} \quad \Omega \setminus \omega, \\
u_{i} = g_0 \quad \text{on} \quad \Gamma, \\
u_{i} = g_1 \quad \text{on} \quad \gamma, \\
u_0 = u(0) \quad \text{in} \quad \Omega \setminus \omega. 
\]

where \( \Omega \) is the whole computational domain and filled with fluid, \( \omega \) is the domain contained in \( \Omega \), \( \Gamma \) is the boundary \( \partial \Omega \), \( \gamma \) is the boundary \( \partial \omega \), \( f \) is external force, \( g_0 \) and \( g_1 \) are boundary conditions. The given function \( f \), \( g_0 \), and \( g_1 \) are defined over \( \Omega \setminus \bar{\omega} \), \( \gamma \) and \( \Gamma \), respectively, and \( u_i \) represents velocity, \( p \) is the pressure, and \( \nu \) represents the inverse of Reynolds number, respectively.

3 A FICTITIOUS DOMAIN METHOD

3.1 FINITE ELEMENT INTERPOLATION

The fractional step method is applied to the Navier-Stokes equation. Flow and pressure fields can be separately solved by the fractional step method as shown in equations (6) and (7). The Crank-Nicolson scheme is employed for the temporal discretization of momentum equation (1). The Galerkin formulation is employed for the spatial discretization. The weighting residual equation is written as follows.

\[
\frac{1}{\Delta t} \int_{\Omega} w_i \tilde{u}_i^{n+1} d\Omega + \frac{1}{2} \int_{\Omega} w_i u_j u_{i,j}^{n+1} d\Omega + \nu \int_{\Omega} w_{i,j} \left( \bar{u}_i^{n+1}_j + \bar{u}_j^{n+1}_i \right) d\Omega = \frac{1}{\Delta t} \int_{\Omega} w_i u_i^n d\Omega - \frac{1}{2} \left( \int_{\Omega} w_{i,j} u_i^{n+1}_j d\Omega + \nu \int_{\Omega} w_{i,j} \left( u_i^{n+1}_j + u_j^{n+1}_i \right) d\Omega \right) + \int_{\Omega} w_i f_i d\Omega + \int_{\Omega} w_{i,i} p^n d\Omega,
\]

\[
\int_{\Omega} w_{i,i} p_i^{n+1} d\Omega = \int_{\Omega} w_{i,i} p_i^n d\Omega - \int_{\Omega} w_{i,i} u_i^n d\Omega,
\]

\[
\frac{1}{\Delta t} \int_{\Omega} w_i u_i^{n+1} d\Omega + \frac{1}{2} \left( \int_{\Omega} w_{i,j} u_i^{n+1}_j d\Omega + \nu \int_{\Omega} w_{i,j} \left( u_i^{n+1}_j + u_j^{n+1}_i \right) d\Omega \right) = \frac{1}{\Delta t} \int_{\Omega} w_i u_i^n d\Omega - \frac{1}{2} \left( \int_{\Omega} w_{i,j} u_i^n d\Omega + \nu \int_{\Omega} w_{i,j} \left( u_i^n + u_j^n \right) d\Omega \right) + \int_{\Omega} w_i f_i d\Omega + \int_{\Omega} w_{i,i} p^n d\Omega.
\]
where \( w_i \) and \( u_i \) are weighting functions. As for the spatial discretization, the bubble function interpolation for the velocity and the linear interpolation for pressure is applied and expressed as follows, for linear interpolation and

\[
p = \Psi_1 p_1 + \Psi_2 p_2 + \Psi_3 p_3, \tag{10}
\]

\[
\Psi_1 = L_1, \Psi_2 = L_2, \Psi_3 = L_3, \tag{11}
\]

for Bubble function interpolation

\[
u_i = \Phi_1 u_{i1} + \Phi_2 u_{i2} + \Phi_3 u_{i3} + \Phi_4 u_{i4}, \tag{13}
\]

\[
\Phi_4 u_{i4} = \frac{1}{3}(u_{i1} + u_{i2} + u_{i3}), \tag{14}
\]

\[
\Phi_1 = L_1, \Phi_2 = L_2, \Phi_3 = L_3, \Phi_4 = L_4, \tag{15}
\]

where \( L_1 \sim L_3 \) the linear interpolation function.

![Fig.2 bubble function interpolation](image)

![Fig.3 linear interpolation](image)

### 3.2 FICTITIOUS DOMAIN FORMULATION

The fictitious domain method is applied to the basic equation, and the finite element interpolation can be written as follows,

\[
\frac{1}{\Delta t} \int_{\Omega} W_i U_i^{n+1} d\Omega + \frac{1}{2} \left\{ \int_{\Omega} W_i U_j^{n+1} U_{i,j} d\Omega + \nu \int_{\Omega} W_{i,j} (U_{i,j}^{n+1} + U_{j,i}^{n+1}) \right\} d\Omega = 0,
\]

\[
\int_{\Omega} W_i f_i d\Omega + \int_{\Omega} W_i p dx + \int_{\omega} W_i \lambda d\Omega = 0, \tag{16}
\]

where \( W_h \) is interpolation function based on the bubble function. In order to integrate easily the last terms in equations (15) added by the fictitious domain method, the following Dirac’s delta function is introduced.

\[
\delta(X - X_i) = \begin{cases} \infty & (X = X_i) \\ 0 & (X \neq X_i). \end{cases} \tag{18}
\]
The integral form is expressed as follows,

\[ \int_{-\infty}^{\infty} \delta(X - X_i) = \begin{cases} 
0 & (X = X_i) \\
1 & (X \neq X_i). 
\end{cases} \] (19)

The interpolations of \( \lambda \) and \( \mu \), which are expressed by \( \lambda_h \) and \( \mu_h \), can be expressed as follows,

\[ \lambda_h = \sum_{i=1}^{Nd} \lambda_i \delta(X - X_i), \] (20)

\[ \mu_h = \sum_{i=1}^{Nd} \mu_i \delta(X - X_i). \] (21)

Then the integration on the domain \( \omega \) can be written as follows,

\[ \int_{\omega} \lambda_h w_h dx = \sum_{i=1}^{Nd} \lambda_i w_h(X_i), \] (22)

\[ \int_{\omega} \mu_h (U_h - g_1) d\omega = \sum_{i=1}^{Nd} \mu_i (U_h(X) - g_1(X_i)). \] (23)

4 NEWTON LAW

4.1 PARTICLE MOTION

The particle motion is caused by the fluid force and the gravity. The motion of the particle is governed by the momentum equation given by the Newton law,

\[ m_p \frac{dv_p}{dt} = (1 - \frac{\rho_f}{\rho_p})m_p g + F_p, \] (24)

\[ I_p \frac{d\omega_p}{dt} = T_p, \] (25)

\[ \frac{dG_p}{dt} = v_p, \] (26)

where \( v_p \) represents the translation velocity, \( \omega_p \) is the angular speed, \( m_p \) is the mass of the particle, \( I_p \) is the moment of inertia, \( G_p \) is central coordinate, \( g \) is the gravity, \( \rho_p \) is the density of the particle, \( F_p \) is the fluid force, \( T_p \) is the moment force imposed on the particle by the fluid, respectively, in which subscripted \( p \) expresses the quantity concerned with the particle. The force and the moment imposed on the particle by the fluid are described as follows,

\[ F_p = \int_{\gamma} \sigma ndx, \] (27)

\[ T_p = \int_{\gamma} (x - G_p) \times (\sigma n) dx, \] (28)

where \( \sigma = -pI + \nu(\nabla u + \nabla u^T) \) is stress tensor, \( x \) is the generic point and \( n \) is outward unit normal on the boundary \( \gamma \).
The velocity and the centroid of the particle are computed as follows,

\[ v_{pn}^{n+1} = v_{pn}^n + \Delta t \left( 1 - \frac{\rho_f}{\rho_p} \right) g + \Delta t \frac{F^w_p}{m_p}, \]  
(29)

\[ w_{pn}^{n+1} = w_{pn}^n + \Delta t \frac{T^n_p}{I_p}, \]  
(30)

\[ G_{pn}^{n+1} = G_{pn}^n + \Delta t v_{pn}^n. \]  
(31)

From the above equations, total velocity of the particle is obtained as follows,

\[ u_{pn}^{n+1} = v_{pn}^{n+1} + w_{pn}^{n+1} \times (x_{pn}^{n+1} - G_{pn}^{n+1}) \] \text{on} \ \gamma. \]  
(32)

### 4.2 COLLISION

In this research, two dimensional analysis of collision between particles and wall are also considered. The extension of the idea to three dimensional analysis is straightforward. To express the equations considering the collision, the following equation is used instead of the momentum equation(7).

\[ m_p \frac{dv_p}{dt} = (1 - \frac{\rho_f}{\rho_p})m_pg + F_p + F^c_p, \]  
(33)

where \(F^c_p\) is an impulsive force imposed on the particle by the other particles or the walls. The impulsive force \(F^c_p\) is defined as follows,

\[ F^c_p = F_w^p + F_p^p, \]  
(34)

where \(F_w^p\) is impulsive force between the particle and the wall and \(F_p^p\) is the impulsive force between particles. The impulsive force is hard to calculate in the direct manner. Therefore in order to calculate the impulsive force, three equations are introduced as follows;

\[ m_p(v_a(t_2) + v_b(t_2)) = m_p(v_a(t_1) + v_b(t_1)), \]  
(35)

\[ e = \frac{v_a(t_2) - v_b(t_2)}{v_a(t_1) - v_b(t_1)}, \]  
(36)

\[ \int_{t_1}^{t_2} F^c_p dt = m_p \int_{t_1}^{t_2} \frac{dv}{dt} dt = m_p v(t_2) - m_p v(t_1). \]  
(37)

where \(e\) is repulsion coefficient.

To judge the touch between the particles or the particle and wall and to calculate the impulsive force, the short range is set between the particles or the wall, because the computational mesh for particle does not allow to overlap each other and to go through the wall.

The collision between the particle and the particle can be modeled as;
\[ t = t_1 \]

![Collision between particle and particle](image)

Fig.4 Collision between particle and particle

\[ F_{p_A} = \begin{cases} 
0 & \text{if } d > (r_a + r_b + \rho), \\
\frac{m_p}{2\pi}((-1 - e)v_a(t_1) + (1 + e)v_b(t_1)) & \text{if } d \leq (r_a + r_b + \rho), 
\end{cases} \tag{38} \]

\[ F_{p_B} = \begin{cases} 
0 & \text{if } d > (r_a + r_b + \rho), \\
\frac{m_p}{2\pi}((1 + e)v_a(t_1) + (-1 - e)v_b(t_1)) & \text{if } d \leq (r_a + r_b + \rho). 
\end{cases} \tag{39} \]

where \( d \) is the distance between the center of the particles, in which \( r_a \) and \( r_b \) are radius of the particle \( A \) and \( B \), respectively, and \( \rho \) is the spare short range. If the particles go into this short range, and the collision between the particles is happened and the impulsive force is obtained from eqs.(21) and (22). Similarly, the collision between the particle and the wall is shown as follows. The collision between the particle and the wall is expressed as:

\[ F_{p_w} = \begin{cases} 
0 & \text{if } d > (r + \rho), \\
\frac{-m_p}{2\pi}v(t_1)(e + 1) & \text{if } d \leq (r + \rho). 
\end{cases} \tag{40} \]

where \( d \) is the distance between the center of the particle and the wall and \( \gamma \) is the radius of the particle.
5 NUMERICAL EXAMPLE

The fictitious domain method is applied to the free falling problem. The simulation of a free falling particles in the closed channel ($\Omega = (-12.0, 12.0) \times (-22.0, 2.0)$) is given as an example. The diameter $d$ of the particle is 1.0 and the position of the particle at $t = 0.0$ is shown in figure 6. Initial velocity of the particle is given $v_p = 0.0$ and angular speed of the particle is given $\omega_p$. The boundary condition of the velocity of fluid is $u = 0.0$ on $\Gamma$. The density of the fluid is $\rho_f = 1.00$ and the density of the all particles are $\rho_p = 2.30$. The velocity of the fluid is $\mu = 0.004$. Time increment is $\Delta t = 0.001$. The repulsion coefficient is $e = 0.3$. 

Fig.6 computational model
The numerical result of the falling particles is shown in figures (7)∼(11).

Fig. 7 $t = 0.1$

Fig. 8 $t = 2.0$

Fig. 9 $t = 4.0$
7 CONCLUSION

The application of the fictitious domain method to the incompressible Navier-Stokes around the particles is discussed. The fictitious domain method is applied to the moving boundary problem. The finite element method based on the mixed interpolation can be applied to the fictitious domain method with the bubble function interpolation for velocity and linear function for pressure. The numerical result show that the fictitious domain method can be applied to the free falling particles model without remeshing technique. The usefulness of the fictitious domain method is shown by the numerical experiments.

REFERENCE


